A Survey of Determinacy and $L(\mathbb{R})$

We survey the development of determinacy theory and the theory of $L(\mathbb{R})$, particularly as it relates to the inductive analysis of the projective sets and beyond.

Let ω^ω be the "reals."

For convenience, we work in "perfect product spaces," that is, products of ω and ω^ω (with at least one factor being ω^ω).

$\mathbf{\Sigma}^0_1$ =open sets, $\quad \mathbf{\Pi}^0_1$ =closed sets.
$\mathbf{\Sigma}^1_{n+1} = \exists^{\omega^\omega}\mathbf{\Pi}^1_n, \quad \mathbf{\Pi}^1_{n+1} = \check{\mathbf{\Sigma}}^1_{n+1} = \forall^{\omega^\omega}\mathbf{\Sigma}^1_n,$

$\mathbf{\Delta}^1_{n+1} = \mathbf{\Sigma}^1_{n+1} \cap \mathbf{\Pi}^1_{n+1}.$

Definition 1. $A \subseteq \omega^\omega$ is κ-Suslin iff there is a tree T on $\omega \times \kappa$ (i.e., $T \subseteq (\omega \times \kappa)^{<\omega}$) such that $A = p[T]$. That is $x \in A \Leftrightarrow \exists f \in \kappa^\omega\ [\forall n\ (x \upharpoonright n, f \upharpoonright n) \in T]$.

Almost by definition, $\mathbf{\Sigma}^1_1 = \omega$-Suslin.

Let WO $\subseteq 2^\omega$ be the standard set of codes of wellorderings of ω. For $x \in$ WO, let $|x|$ denote the length of the wellordering coded by x. Let WF be the set of codes of wellfounded relations on ω.

Let us begin by considering ω_1.

If $\lambda \leq \kappa$, we say $\kappa \to (\kappa)^\lambda$ if for all partitions $\mathcal{P}$ of κ^λ (the increasing functions from λ to κ) into two pieces, there is a homogeneous set of size κ. We say $\kappa \to \kappa^{<\lambda}$ if for all $\rho < \lambda$ we have $\kappa \to \kappa^\rho$.

We say κ has the strong partition property if $\kappa \to \kappa^\kappa$, and κ has the weak partition property if $\kappa \to \kappa^{<\kappa}$.

Definition 2. $f\colon \kappa \to \mathrm{ON}$ is of the *correct type* if it is increasing, discontinuous, and of uniform cofinality ω, that is, there is an $f'\colon \omega \cdot \kappa \to \mathrm{ON}$ such that $f(\alpha) = \sup_{\beta < \omega \cdot (\alpha+1)} f'(\beta)$.

We say $\kappa \xrightarrow{cub} (\kappa)^\lambda$ if for every partition $\mathcal{P}$ of the functions $f\colon \lambda \to \kappa$ of the correct type into two pieces, there is a c.u.b. $C \subseteq \kappa$ which is homogeneous for $\mathcal{P}$.

Fact 1.

$$\kappa \xrightarrow{cub} (\kappa)^\lambda \Rightarrow \kappa \to (\kappa)^\lambda$$

$$\kappa \to (\kappa)^{\omega \cdot \lambda} \Rightarrow \kappa \xrightarrow{cub} (\kappa)^\lambda$$

Theorem 1. $\omega_1 \to (\omega_1)^{<\omega_1}$.

Proof. Will show $\omega_1 \to (\omega_1)^\omega$; general case similar. Identify each real x with an ω^2 sequence $\{x_\alpha\}_{\alpha < \omega^2}$ by $x_{\omega \cdot n + m}(p) = x(\langle n, m, p \rangle)$, where $(n, m, p) \to \langle n, m, p \rangle$ is a reasonable bijection.

Let $\mathcal{P}$ be a partition of the functions $f\colon \omega \to \omega_1$ of the correct type into $\{0,1\}$.

Consider the integer game where I, II play out x, y respectively. If there is an $\alpha < \omega^2$ such that x_α or y_α is not in WO, then I wins iff for the least such α we have $x_\alpha \in \mathrm{WO}$. If for all $\alpha < \omega^2$, x_α, $y_\alpha \in \mathrm{WO}$, then let $F(n) = \sup\{\max(|x|_{\omega\cdot n+m}, |y|_{\omega\cdot n+m})\}$. II wins iff $\mathcal{P}(F) = 1$.

Suppose τ is winning for II. For $\alpha < \omega^2$, $\beta < \omega_1$, let

$$x \in R_{\alpha,\beta} \Leftrightarrow \forall \alpha' \le \alpha \; (x_{\alpha'} \in \mathrm{WO} \wedge |x_{\alpha'}| \le \beta).$$

Each $R_{\alpha,\beta} \in \mathbf{\Delta}^1_1$. Let

$$x \in R_\alpha \Leftrightarrow \forall \alpha' \le \alpha \; (x_{\alpha'} \in \mathrm{WO}).$$

If $x \in R_\alpha$, then $\tau(x) \in R_\alpha$. Let

$$h(\alpha,\beta) = \sup\{|\tau(x)_\alpha| \colon x \in R_{\alpha,\beta}\},$$

which exists since $\tau[R_{\alpha,\beta}] \in \mathbf{\Sigma}^1_1$.

Let $C \subseteq \omega_1$ be a c.u.b. set closed under h.

Suppose $f\colon \omega \to C$ is of the correct type (i.e., increasing). Let $f'\colon \omega^2 \to \omega_1$ be such that $f(n) = \sup_m f'(\omega\cdot n+m)$. Let x code f'. Let $y = \tau(x)$. Let $g'(\omega\cdot m+n) = |y_{\omega\cdot n+m}|$. Thus, $g'(\omega\cdot n+m) < f(n)$. Then $F = f$ and so $\mathcal{P}(f) = \mathcal{P}(F) = 1$ as τ is winning. $\square$

The Kunen Tree

Theorem 2. *(Kunen) There is a tree T on $\omega\times\omega_1$ such that for all $f\colon \omega_1 \to \omega_1$, there is an $x \in \omega^\omega$ such that T_x is wellfounded and $\forall\alpha \geq \omega$ $|T_x \restriction \alpha| > f(\alpha)$.*

Proof. Let U be a tree on $\omega \times \omega_1$ projecting to WF such that for all $x \in$ WF, $U_x \restriction |x|$ is illfounded.

Let S be a tree on $\omega \times \omega$ projecting to a $\mathbf{\Sigma}^1_1$ complete set. Thus, $\sup\{|S_y|\colon S_y \text{ is wellfounded }\} = \omega_1$.

Define V on $\omega \times \omega \times \omega_1 \times \omega \times \omega$ as follows:

$$\begin{aligned}(s, a, \vec{\alpha}, b, c) \in V \leftrightarrow{}& (a, \vec{\alpha}) \in U \wedge (b, c) \in S \\ &\wedge \exists\sigma \text{ extending } s\ (\sigma(a) = b)\end{aligned}$$

Suppose $f\colon \omega_1 \to \omega_1$. Consider the Solovay game where I, II play out x, y, and II wins iff

$$x \in \mathrm{WF} \to (S_y \text{ is wellfounded } \wedge |S_y| > f(|x|)).$$

By boundedness and property of S, II wins. Let σ be winning for II. Then V_σ is wellfounded and for all $\alpha \geq \omega$, $|V_\sigma \restriction \alpha| > f(\alpha)$.

Finally, to get T, weave the 2^{nd}, 3^{rd}, 4^{th}, and 5^{th} coordinates into a single coordinate. $\square$

Note: The Kunen tree is Δ^1_1 in the codes for countable ordinals.

Note: Can also prove a version using linear orders instead of trees. That is, there is a map $s \to T(s)$ assigning to each $s \in \omega^{<\omega}$ a wellordering of ω_1 such that if $s \subseteq t$, then $T(s) \subseteq T(t)$. For any $f : \omega_1 \to \omega_1$ there is an x such that $T(x) = \bigcup_n T(x \restriction n)$ is a wellordering and for all $\alpha \geq \omega$, $|T(x) \restriction \alpha) > f(\alpha)$.

Two Applications of the Kunen Tree

For the first application, we sketch how to analyze the subsets of ω_1 and get the strong partition relation on ω_1.

- Strong partition on ω_1 due to Martin. Martin's original proof used indiscernibles for $L(x)$ to code the functions.
- Martin's proof provided a framework for proving partition relations in general.
- Recently, a new proof has been found. Four proofs now known.

Idea of Kunen: If you can analyze the measures on κ, then you can analyze the subsets of κ.

Let ν be a (non-principal) measure on ω_1. Let $f\colon \omega_1 \to \omega_1$ satisfy:

(1) There is a ν measure one set A such that if $\alpha \leq \beta$ are in A, then $f(\alpha) \leq f(\beta)$.
(2) There is no measure one set on which f is constant.
(3) If $[f']_\nu < [f]_\nu$, then f' does not satisfy (1) and (2).

Clearly f exists, as the identity satisfies (1) and (2). Then $f(\nu) = W_1^1 =$ the normal measure on ω_1. Fix A as in (1). Define $g(\alpha) = \sup\{\beta \in A\colon f(\beta) \leq \alpha\}$. So $g\colon \omega_1 \to \omega_1$ using (1).

Fix x so that T_x is wellfounded and for all α ($\geq \omega$) $|T_x \restriction \alpha| > g(\alpha)$. So, for $\alpha \in A$, $|T_x \restriction f(\alpha)| > \alpha$.

For $\alpha \in A$, let $h(\alpha)$ be the unique $\beta < f(\alpha)$ such that $|(T_x \restriction \alpha)(\beta)| = \alpha$.

Let $\nu_1 = h_1(\nu)$. Let $f = f_1$, $g = g_1$, $h = h_1$, $x = x_1$. Continue to define f_n, g_n, h_n, x_n, ν_{n+1}. The sequence $h_1, h_2 \circ h_1, \ldots$ is ν decreasing, so this process terminates after finitely many steps in a principal measure ν_{n+1} (say the principal measure on α_0).

If we let w_i be the wellordering $T_{x_{n-i+1}}$, we may summarize this as follows. Let $l = l(w_1, \ldots, w_n, \alpha_0)\colon \omega_1^n \to \omega_1$ be defined by:

$$\begin{aligned}
l(\beta_1, \ldots, \beta_n) &= |(w_n \restriction \beta_n)(l(\beta_1, \ldots, \beta_{n-1})| \\
l(\beta_1, \ldots, \beta_{n-1}) &= |(w_{n-1} \restriction \beta_{n-1})(l(\beta_1, \ldots, \beta_{n-2})| \\
&\vdots \\
l(\beta_1) &= |(w_1 \restriction \beta_1)(\alpha_0)|
\end{aligned}$$

Then we have:

Theorem 3. *Let ν be a measure on ω_1. Then there are wellorderings $w_1, \ldots, w_n$ of ω_1 and an $\alpha_0 < \omega_1$ such that*

$$\nu(A) = 1 \Leftrightarrow \exists c.u.b.\ C \subseteq \omega_1\ \forall \beta_1, \ldots, \beta_n \in C\ [l(\beta_1, \ldots, \beta_n) \in A].$$

Corollary 1. *Every measure on ω_1 is equivalent to a product of the normal measure ($\nu \approx W_1^n$).*

From this analysis of measures and Kunen's idea, we get an analysis of the subsets of ω_1. Every subset of ω_1 is a countable union of "simple sets." A simple set is one of the form $S = l[C^n]$ where $l = l(w_1, \ldots, w_m, \alpha_0)$ is as above, and C is a c.u.b. set (actually, a codeable c.u.b. set).

Via Martin's method, this proves the strong partition relation on ω_1.

- A slight modification of the above allows us to code functions instead of sets (need only modify the first step).

- The sets $R_{\alpha,\beta}$ again are $\mathbf{\Delta}^1_1$. Also, any $\mathbf{\Sigma}^1_1$ subset of R_α takes boundedly many values at α.

Martin and Paris showed ω_2 has the weak but not the strong partition property (will return to this later).

Second Application of the Kunen Tree

Let ν denote the supercompactness measure (i.e., fine, normal) on $\mathcal{P}_{\omega_1}(\kappa)$ for λ a cardinal.

- Solovay showed the existence of the measure from $\mathrm{AD}^{\mathbb{R}}$, for $\lambda < \Theta$.
- Hypothesis reduced to AD for λ within scales by results of Harrington, Kechris.
- Martin, Steel showed $\mathbf{\Sigma}^2_1$ has the scale property in $L(\mathbb{R})$.
- Woodin showed the supercompactness measure on $\mathcal{P}_{\omega_1}(\kappa)$ is unique.

Theorem 4. *Let $C \subseteq \omega_1$ be c.u.b. Let λ be a cardinal and assume the supercompactness measure on $\mathcal{P}_{\omega_1}(\kappa)$ exists. Then $\forall^* S \in \mathcal{P}_{\omega_1}(\kappa)$ $[o.t.(S) \in C]$.*

Proof. Fix a c.u.b. $C \subseteq \omega_1$, and suppose the theorem fails. Fix a wellordering $<_w$ of ω_1 such that for all infinite α, $|<_w \restriction \alpha| > N_C(\alpha)$, where $N_C(\alpha)$ denotes the next element in C after α.

Fix $\rho < \lambda$ such that for ν almost all $S \in \mathcal{P}_{\omega_1}(\kappa)$, $\pi(S \cap \rho)$ is the largest element of C below $\pi(S)$. Here π denotes the transitive collapse.

Define a wellordering $\prec$ of ρ as follows.

$$\alpha \prec \beta \Leftrightarrow \forall^*_\nu S \ [\pi(\alpha) <_w \pi(\beta)]$$

Let $\rho' < \lambda$ be the length of $\prec$, and let $h\colon \rho \to \rho'$ be the corresponding bijection.

For ν almost all S we have:
(1) S is closed under h.
(2) $\forall \alpha, \beta \in S \cap \rho \ [\alpha \prec \beta$ iff $\pi_S(\alpha) <_w \pi_S(\beta)]$.

For S satisfying (1) and (2) we have

$$\text{o.t.}(S \cap \rho') \geq | <_w \restriction \pi(S \cap \rho) | > N_C(\pi(S \cap \rho)).$$

This contradicts the definition of ρ. $\square$

Scale theory of the projective sets.

Moschovakis introduced the notion of a *scale* on a pointset A. A semi-scale on A is a collection of norms $\{\phi_n\}_{n\in\omega}$ on A (i.e., $\phi_n\colon A \to \mathrm{ON}$) satisfying:

If $\{x_m\}_{m\in\omega} \subseteq A$ converges to x, and for each n the values $\phi_n(x_m)$ are eventually constant, then $x \in A$. $\{\phi_n\}_{n\in\omega}$ is a scale if it also satisfies the additional lower semi-continuity property:

$$\forall n\ \phi_n(x) \leq \lambda_n \doteq \lim_{m\to\infty} \phi_n(x_m).$$

We say $\{\phi_n\}$ is an α-semi-scale, or α-scale if all the norms map into α.

Fact 2. For all cardinals κ, A is κ-Suslin iff A has a κ-semi-scale iff A has a κ-scale.

We say $\{\phi_n\}$ is a $\boldsymbol{\Gamma}$-scale if all of the ϕ_n are $\boldsymbol{\Gamma}$-norms. That is, there are $\boldsymbol{\Gamma}$, $\check{\boldsymbol{\Gamma}}$ relations $\leq_n^{\boldsymbol{\Gamma}}$, $\leq_n^{\check{\boldsymbol{\Gamma}}}$ such that for all $y \in A$:

$$\forall x\, [(x \in A \wedge \phi_n(x) \leq \phi_n(y)) \leftrightarrow x \leq_n^{\boldsymbol{\Gamma}} y \leftrightarrow x \leq_n^{\check{\boldsymbol{\Gamma}}} y].$$

Definition 3. $\mathbf{\Gamma}$ has the prewellordering property if every $\mathbf{\Gamma}$ set admits a $\mathbf{\Gamma}$ norm. $\mathbf{\Gamma}$ has the scale property if every $\mathbf{\Gamma}$ set admits a $\mathbf{\Gamma}$-scale.

Martin-Moschovakis, and Moschovakis proved the first and second *periodicity theorems*.

Theorem 5. *(second periodicity) Assume $\mathbf{\Gamma}$ is closed under $\wedge$, $\vee$, $\exists^{\omega^\omega}$ and has the scale property. If $\mathbf{\Delta}$-determinacy holds, then $\forall^{\omega^\omega}\mathbf{\Gamma}$ has the scale property.*

Thus, assuming PD, $\mathbf{\Pi}^1_{2n+1}$, $\mathbf{\Sigma}^1_{2n+2}$ have the scale property.

We summarize the basic "scale theory" of the projective sets.

Definition 4. (Moschovakis) $\boldsymbol{\delta}^1_n$ = the supremum of the lengths of the $\mathbf{\Delta}^1_n$ prewellorderings of ω^ω.

Fact: If ϕ is a $\mathbf{\Pi}^1_{2n+1}$ (regular) norm on a $\mathbf{\Pi}^1_{2n+1}$-complete set, then ϕ is onto $\boldsymbol{\delta}^1_{2n+1}$.

$\mathbf{\Pi}^1_{2n+1}$ admits a $\mathbf{\Pi}^1_{2n+1}$-scale on $\boldsymbol{\delta}^1_{2n+1}$, and nothing smaller.

From this and the Moschovakis coding lemma it follows that

$$\boldsymbol{\delta}^1_{2n+1}\text{-Suslin} = \mathbf{\Sigma}^1_{2n+2}.$$

Recall the Kunen-Martin theorem:

Theorem 6. *If $\prec$ is a κ-Suslin wellfounded relation on ω^ω, then $|\prec| < \kappa^+$.*

From this and the coding lemma it follows that $\boldsymbol{\delta}^1_{2n+2} = (\boldsymbol{\delta}^1_{2n+1})^+$.

From scale($\mathbf{\Pi}^1_{2n+1}$) we have every $\mathbf{\Pi}^1_{2n}$ set admits a $\mathbf{\Delta}^1_{2n+1}$-scale, and hence every $\mathbf{\Pi}^1_{2n}$, $\mathbf{\Sigma}^1_{2n+1}$ set is λ_{2n+1}-Suslin for some $\lambda_{2n+1} < \boldsymbol{\delta}^1_{2n+1}$. From the Kunen-Martin theorem $\lambda^+_{2n+1} = \boldsymbol{\delta}^1_{2n+1}$.

We must have $\mathrm{cof}(\lambda_{2n+1}) = \omega$. Otherwise every $\mathbf{\Sigma}^1_{2n+1}$ set would be a $< \boldsymbol{\delta}^1_{2n+1}$ union of $\mathbf{\Sigma}^1_{2n}$ sets. This contradicts the following.

Theorem 7. *(Martin) $\mathbf{\Delta}^1_{2n+1}$ is closed under $< \boldsymbol{\delta}^1_{2n+1}$ length unions and intersections.*

We summarize the general properties of the $\boldsymbol{\delta}^1_n$:

$\boldsymbol{\delta}^1_1 = \omega_1, \quad \boldsymbol{\delta}^1_2 = \omega_2$ (Martin)

$\boldsymbol{\delta}^1_{2n+2} = (\boldsymbol{\delta}^1_{2n+1})^+$ (Kunen, Martin)

$\boldsymbol{\delta}^1_{2n+1} = \lambda^+_{2n+1}, \quad \mathrm{cof}(\lambda_{2n+1}) = \omega.$ (Kechris)

$\boldsymbol{\delta}^1_{2n+1} \to (\boldsymbol{\delta}^1_{2n+1})^\omega$ (Martin) $\quad \boldsymbol{\delta}^1_{2n+2} \to (\boldsymbol{\delta}^1_{2n+2})^\omega$ (Kunen)

Martin-Solovay showed $\boldsymbol{\delta}^1_3 = \omega_{\omega+1}$, so by Kunen-Martin $\boldsymbol{\delta}^1_4 = \omega_{\omega+2}$.

Steel, building on work of Martin-Steel, developed a general scale theory of $L(\mathbb{R})$. This identifies the scaled pointclasses and Suslin cardinals in $L(\mathbb{R})$.

Homogeneous Trees

Kunen and Martin independently discovered special cases of the homogeneous tree construction; Kechris formulated the general notion.

Definition 5. A tree T on $\omega \times \kappa$ is homogeneous if there are measures μ_s for $s \in \omega^{<\omega}$ satisfying:

(1) $\mu_s(T_s) = 1$
(2) If t extends s, then $\pi_{s,t}(\mu_t) = \mu_s$, where $\pi_{s,t}$ is the natural projection map from T_t to T_s.

(3) If T_x is illfounded and $A_1, A_2, \ldots$ are given with $\nu_{x\restriction n}(A_n) = 1$, then $\exists f \in \kappa^\omega\ [(x,f) \in [T] \wedge \forall n\ (f \restriction n \in A_n)]$.

(3) is equivalent to saying the direct limit of the ultrapowers by the $\mu_{x\restriction n}$ measures is wellfounded.

Kunen proposed a program for computing the $\boldsymbol{\delta}^1_{2n+1}$ bases on the homogeneous tree construction. For example,

$$\lambda_5 = \sup j_\nu(\boldsymbol{\delta}^1_3)$$

where the supremum ranges over measures in the homogeneous tree on a $\boldsymbol{\Pi}^1_3$-complete set.

Kunen also showed there is a $\boldsymbol{\Delta}^1_3$ coding of the subsets of $\omega_\omega = \lambda_3$. That is, there is a map $\pi\colon \omega^\omega \overset{\text{onto}}{\to} \mathcal{P}(\omega_\omega)$ such that for every $\alpha < \omega_\omega$, $\{x\colon \alpha \in \pi(x)\} \in \boldsymbol{\Delta}^1_3$.

<u>The Martin Tree</u>

Theorem 8. *(Martin) There is a tree T on $\omega \times \boldsymbol{\delta}^1_3$ such that for all $f\colon \boldsymbol{\delta}^1_3 \to \boldsymbol{\delta}^1_3$, there is a z with T_z wellfounded and a c.u.b. $C \subseteq \boldsymbol{\delta}^1_3$ such that for all $\alpha \in C$,*

$$f(\alpha) < |T_z \restriction \sup_\nu j_\nu(\alpha)|,$$

where

- *If cof*$(\alpha) = \omega$, *we use* $|T_z \restriction \alpha|$.
- *If cof*$(\alpha) = \omega_1$, *the supremum ranges over measures* ν *occuring in the homogeneous tree on a* $\mathbf{\Pi}^1_1$*-complete set (i.e., the measures* W_1^m*).*
- *If cof*$(\alpha) = \omega_2$,*the supremum ranges over measures* ν *occuring in the homogeneous tree on a* $\mathbf{\Pi}^1_2$*-complete set.*

Theorem follows as for the Kunen tree given the following:

Theorem 9. *There is a* $\mathbf{\Pi}^1_3$*-complete set* P, *a* $\mathbf{\Pi}^1_3$ *norm* $z \to |z|$ *from* P *onto* $\boldsymbol{\delta}^1_3$ *and a homogeneous tree* S *on* $\omega \times \boldsymbol{\delta}^1_3$ *with* $P = p[S]$ *satisfying the following. There is a c.u.b.* $C \subseteq \boldsymbol{\delta}^1_3$ *such that for all* $\alpha \in C$ *there is a* $z \in P$ *with* $|z| = \alpha$ *and* $S_z \restriction \sup_\nu j_\nu(\alpha)$ *illfounded.*

Furthermore, if $(z, \vec{\beta}) \in [S]$, *then* $|z| \leq \beta_0$.

We sketch the proof, abstracting the hypotheses.

We assume:

(1) There is a $\mathbf{\Delta}^1_3$ coding of the subsets of $\boldsymbol{\delta}^1_3$.
(2) $\boldsymbol{\delta}^1_3$ has the weak partition relation (only need to get homogeneity of S).
(3) for every $\alpha < \boldsymbol{\delta}^1_3$ and measure ν on λ_3, $j_\nu(\alpha) < \boldsymbol{\delta}^1_3$.

(4) There is a $\mathbf{\Pi}^1_2$ set B_2 and a norm $x \to |x|_2 < \omega_2$ from B_2 onto ω_2 satisfying: ($\star$) There is a homogeneous tree U on $\omega \times \omega_\omega$ with $B_2 = p[U]$ such that for all $x \in B_2$, $|x|_2 \leq \psi_0(x) < \omega_2$, where $\{\psi_n\}_{n\in\omega}$ denotes the semi-scale from the left-most branch.

It follows from (1) that there is a $\mathbf{\Delta}^1_3$ set B_ω and a $\mathbf{\Delta}^1_3$ norm $x \to |x|_\omega$ on B_ω.
Assume $\pi(z) \subseteq \omega_2 \times \omega_\omega \times \omega_\omega$.

1) We define P and the $\mathbf{\Pi}^1_3$-norm on it.

For $\alpha < \omega_2$, let $A^\alpha_z = \{(\beta,\gamma)\colon (\alpha,\beta,\gamma) \in \pi(z)\}$.
Define $P(z) \leftrightarrow \forall \alpha < \omega_2\ A^\alpha_z$ is wellfounded.
$P \in \mathbf{\Pi}^1_3$. For $z \in P$, let $|z| = \sup_{\alpha<\omega_2} |A^\alpha_z|$.

2) We define the homogeneous tree V.

Define

$$\begin{aligned} R(z, x_1, x_2, x_3) \leftrightarrow\ & x_1 \in B_2 \wedge x_2, x_3 \in B_\omega \\ & \wedge (|x_1|_2, |x_2|_\omega, |x_3|_\omega) \in \pi(z). \end{aligned}$$

$R \in \mathbf{\Delta}^1_3$, so let $R_2 \in \mathbf{\Pi}^1_2$ project to R, and let T_2 be a homogeneous tree on $\omega^5 \times \omega_\omega$ projecting to T_2.
Define V on $\omega^2 \times \omega_\omega \times \omega \times \omega_\omega$ as in the Kunen-Martin theorem for R:
$(p, s, \vec{\alpha}, u, \vec{\beta}) \in V$ iff $(s, \vec{\alpha}) \in U$ and each $u(i)$ codes more and more of reals u_i, w_i. Each $u(j+1)$

extends the array coded by $u(j)$ by adding one extra pair $(u_i(k), u_{i+1}(k))$ for some i, k, and one extra value $w_i(k)$. For each i, k, if the pairs

$$(u_i(0), u_{i+1}(0)), \ldots, (u_i(k-1), u_{i+1}(k-1))$$

are added at stages $q_0, \ldots, q_{k-1} < n = \mathrm{lh}(u)$, then

$$(p \restriction k, s \restriction k, u_i \restriction k, u_{i+1} \restriction k, w_i \restriction k, (\beta_{q_0}, \ldots, \beta_{q_{k-1}}) \in T_2.$$

V is homogeneous.

Apply homogeneous tree construction to V to get a tree S on $\omega \times \boldsymbol{\delta}^1_3$ such that V_z is wellfounded iff S_z is illfounded.

When we identify tuples $(s(0), \alpha_0, u(0), \beta_0)$ with an ordinal in defining the Kleene-Brouwer order $\prec_{V_p}$ on V_p, we order by α_0 first.

Briefly, $(p, \vec{\gamma}) \in S$ iff there is an $f\colon \prec_{V_p} \to \boldsymbol{\delta}^1_3$ order-preserving of the correct type which represents $\vec{\gamma}$.

We define the c.u.b. $C \subseteq \boldsymbol{\delta}^1_3$ as required.

For $\alpha < \omega_2$, $\beta < \boldsymbol{\delta}^1_3$, let

$$P_{\alpha,\beta} = \{z\colon \forall \alpha' \leq \alpha\; A^{\alpha'}_z \text{ is wellfounded of rank } \leq \beta\}.$$

$P_{\alpha,\beta} \in \mathbf{\Delta}^1_3$. If $z \in P_{\alpha,\beta}$, then V_z restricted to tuples $(s, \vec{\alpha}, u, \vec{\beta})$ with $\alpha_0 \leq \alpha$ is wellfounded. This uses property $(\star)$ of U.

Since $P_{\alpha,\beta}$ is ω_ω-Suslin, a tree argument shows that

$$b(\alpha, \beta) \doteq \sup\{|\alpha|_{V_z} \colon z \in P_{\alpha,\beta}\} < \boldsymbol{\delta}^1_3,$$

where $|\alpha|_{V_z}$ denotes the supremum of the ranks of $(s(0), \alpha, u(0), \beta_0)$ in the Kleene-Brouwer order of V_z.

Let C be c.u.b. closed under b.

To see this works, fix $\delta \in C$ with $\mathrm{cof}(\delta) = \omega_2$. Fix $h \colon \omega_2 \to \delta$ increasing and cofinal. Let z be such that for all $\alpha < \omega_2$, A^α_z is wellfounded of rank $h(\alpha)$.

So, $P(z)$ and $\forall \alpha < \omega_2\ P_{\alpha,h(\alpha)}(z)$. Thus, $|\alpha|_{V_z} \leq b(\alpha, h(\alpha)) < \delta$. Hence there is an order-preserving map f from the Kleene-Brouwer order of V_z to δ. If f represents $\vec{\gamma}$, then $\gamma_0 = \delta$, and $\gamma_n \leq \sup_\nu(j_\nu(\delta))$, where ν ranges over the homogeneity measures for U. Λ

Descriptions and the $\boldsymbol{\delta}^1_n$

We illustrate the general ideas by a simple example.

Let μ_2 denote the ω_2-cofinal normal measure on $\boldsymbol{\delta}^1_3$. This is generated by the weak partition relation on $\boldsymbol{\delta}^1_3$, functions $f\colon \omega_2 \to \boldsymbol{\delta}^1_3$ of continuous type, and the ω-cofinal normal measure on ω_2.

Let S^1_1 denote the ω-cofinal normal measure on ω_2. This is generated by the strong partition relation on ω_1, functions $h\colon \omega_1 \to \omega_1$ of the correct type, and the normal measure on ω_1.

We define a μ_2 equivalence class $[F]_{\mu_2}$. Fix $f\colon \omega_2 \to \boldsymbol{\delta}^1_3$ increasing, continuous. We define $F([f]_{S^1_1})$.

We let $F([f])$ be represented w.r.t. S^1_1 by the function which assign to $[h_1]_{W^1_1}$ the value $(f; h_1)$.

$(f; h_1)$ is represented with respect to S^1_1 by the function which assigns to $[h_2]_{W^1_1}$ the value $(f; h_1, h_2)$.

$$(f; h_1, h_2) = f([h_1 \circ h_2]_{W^1_1}).$$

Note: This is well-defined, but the functions must be composed in this order.

In general, we have a collection of hereditarily finite objects d called descriptions, and a way of generating equivalence classes from them (with respect to measures S_1^m on ω_{m+1}. These ordinals are denoted $(d; K_1, \ldots, K_t)$.

In the above example, $K_1 = K_2 = S_1^1$.

Given a function $g\colon \boldsymbol{\delta}_3^1 \to \boldsymbol{\delta}_3^1$, we have a way of "lifting up" to get $(g; d; K_1, \ldots, K_t)$. When $g = id$ this reduces to $(d; K_1, \ldots, K_t)$.

General analysis proceeds in two steps.

1) First show that if $F < (id; d; K_1, \ldots, K_t)$, then for some g, $F < (g; \mathcal{L}(d); K_1, \ldots, K_t)$, where $\mathcal{L}$ is a lowering operator on the descriptions. This step does not involve Martin's theorem.

2) Second, show that

$(g; d'; K_1, \ldots, K_t) < [\sup(id; d'; K_1, \ldots, K_t, K_{t+1})]^+$. This uses Martin's theorem.

We illustrate with the above example.

Let d_0 be the description corresponding to $f([h_1 \circ h_2])$.

Let d_1 correspond to $f([h_2])$.

Let d_2 correspond to $f([h_1])$.

Then $d_0 > d_1 > d_2$ is an $\mathcal{L}$ chain. Note in this simple example we don't increase the measure sequence. If $F < (d_2; K_1, K_2)$, then F is constant μ_2 almost everywhere. So $[(d_2; K_1, K_2)]_{\mu_2} = \boldsymbol{\delta}^1_3 = \omega_{\omega+1}$. So, $[(d_0; K_1, K_2)] = \omega_{\omega+3}$.

<u>Second example</u>:

We compute $j_{\mu_2}(\boldsymbol{\delta}^1_3)$.

Let S^m_1 be the measure on ω_{m+1} defined as follows. Let $<_m$ be the order on ω^m_1 defined by:

$$(\alpha_1, \ldots, \alpha_m) <_m (\beta_1, \ldots, \beta_m) \leftrightarrow$$
$$(\alpha_m, \alpha_1, \alpha_2, \ldots, \alpha_{m-1}) <_{\text{lex}} (\beta_m, \beta_1, \beta_2, \ldots, \beta_{m-1}).$$

S^m_1 is the measure induced by functions $h\colon <_m \to \omega_1$ of the correct type (order-preserving, discontinuous, of uniform cofinality ω) and the measure W^m_1.

Let $\kappa = j_{\mu_2}(\boldsymbol{\delta}^1_3)$. By Martin's theorem, $\kappa = \kappa_1^+$ where

$$\kappa_1 = \sup_m (d; S^m_1),$$

where d corresponds to the function $F(f, h_1) = \sup(f)$. That is, $F(f) = j_{S^m_1}(\sup(f))$.

Let $\rho_m = (d; S_1^m)$. The next lower description is d', where $(d'; S_1^m)$ corresponds to $F(f; h_1) = f([h_1(1)])$. Here, $h_1(1)$ is defined by:

$$h_1(1)(\alpha) = \sup_{\alpha_1 < \cdots < \alpha_{m-1}} h_1(\alpha_1, \ldots, \alpha_{m-1}, \alpha).$$

In other words, if $F < (d; S_1^m)$, then there is a $g\colon \boldsymbol{\delta}_3^1 \to \boldsymbol{\delta}_3^1$ such that $F < (g; d'; S_1^m)$.

By Martin's theorem,

$$\rho_m = [\sup_{n_1} \zeta_{n_1}]^+,$$

where $\zeta_{n_1} = (d^{n_1}; S_1^m, W_1^{n_1})$.

For $1 \leq j_1 \leq n_1$, let d^{j_1} correspond to the the function $F(f; h_1, \vec{\alpha}_1) = f([\beta \to h_1(2)(\vec{\alpha}_1(j_1), \beta)])$.

In general, for $1 \leq k \leq m-1$ let $(d^{j_1,\ldots,j_k}; S_1^m, W_1^{n_1}, \ldots, W_1^{n_k})$ correspond to $F(f, \vec{\alpha}_1, \ldots, \vec{\alpha}_k) = f([h])$ where

$$h(\beta) = h_1(k+1)(\vec{\alpha}_1(j_1), \ldots, \vec{\alpha}_k(j_k), \beta).$$

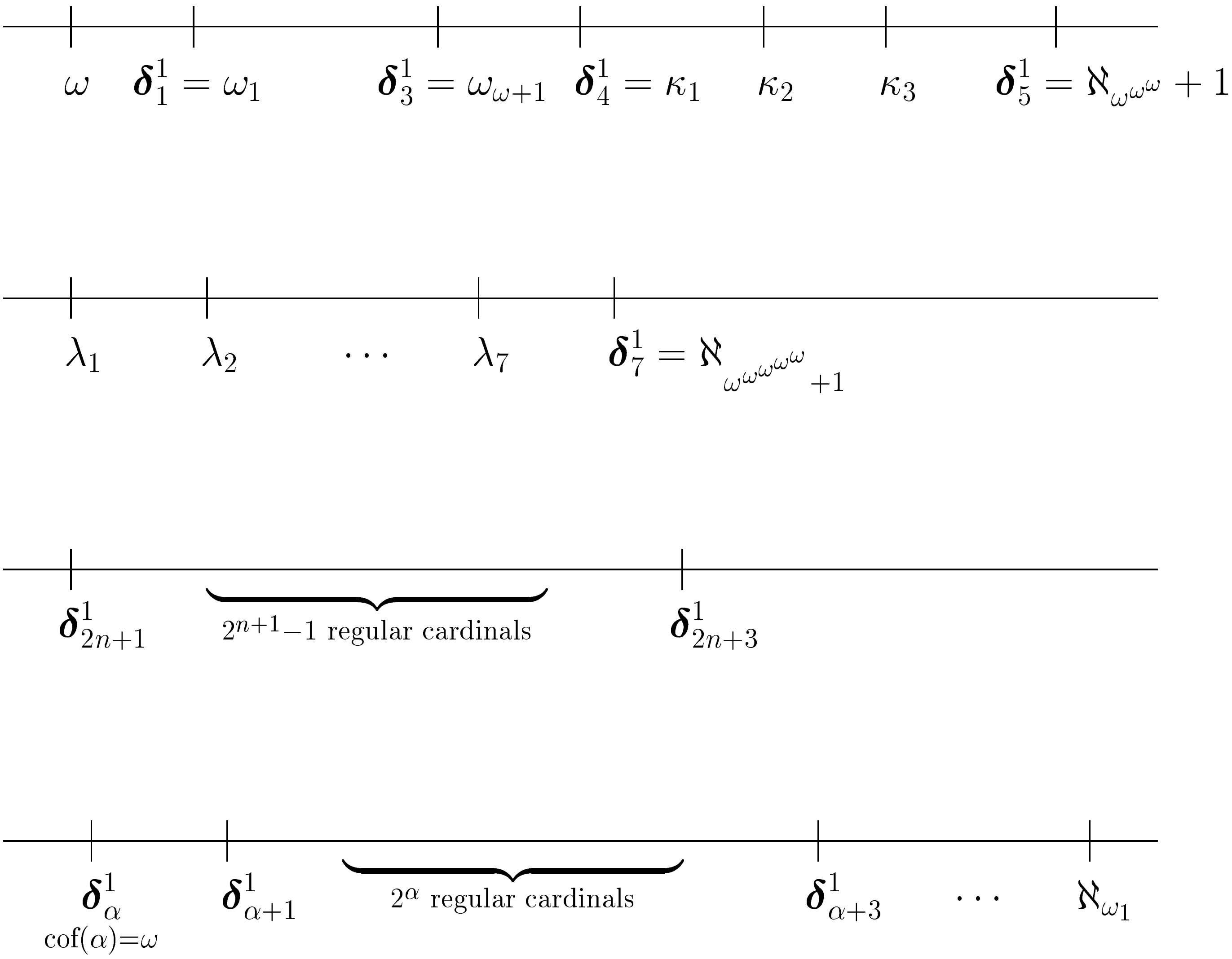
Counting now shows

$$(d^{j_1,\ldots,j_k}; S_1^m, W_1^{n_1}, \ldots, W_1^{n_k}) \leq \aleph_{\omega^{m-2}\cdot(j_1-1)+\omega^{m-3}\cdot(j_2-1)+\cdots+\omega^{m-1-k}\cdot(j_k-2)+1}$$

In particular, $\rho_m \le \aleph_{\omega+1+\omega^{m-1}+1}$. In fact, equality holds.

Thus, $j_{\mu_2}(\boldsymbol{\delta}^1_3) = \aleph_{\omega^\omega+1}$.

We have the following picture.



$$\kappa_1 = \aleph_{\omega+1} \qquad \kappa_2 = \aleph_{\omega\cdot 2+1} \qquad \kappa_3 = \aleph_{\omega^\omega+1}$$

$$\lambda_1 = \aleph_{\omega^{\omega^\omega}+2} \quad \lambda_2 = \aleph_{\omega^{\omega^\omega}+\omega+1} \quad \lambda_3 = \aleph_{\omega^{\omega^\omega}+\omega^\omega+1}$$
$$\lambda_4 = \aleph_{\omega^{\omega^\omega}\cdot 2+1} \quad \lambda_5 = \aleph_{\omega^{\omega^{\omega+1}}+1} \quad \lambda_6 = \aleph_{\omega^{\omega^{\omega}\cdot 2}+1}$$
$$\lambda_7 = \aleph_{\omega^{\omega^{\omega^\omega}}+1}$$

Some properties of the κ_i's (and λ_i's, etc.).

$\kappa_i \to (\kappa_i)^{\boldsymbol{\delta}^1_3}$, but $\kappa_i \not\to (\kappa_i)^{\boldsymbol{\delta}^1_4}$. In particular, they are all measurable.

$\boldsymbol{\delta}^1_4$ is $\boldsymbol{\delta}^2_1$-supercompact, but κ_2, κ_3 are not λ_5-supercompact.

Question: What is the least λ so that κ_i is not λ-supercompact?

Type 1 vs. Type 2 Theories

Type 1: sets of ordinals, measures.
Type 2: sets of sets of ordinals, supercompactness measures, λ regularity.

Definition 6. (Becker) κ is λ-regular if whenever $\alpha < \kappa$ and $f\colon \alpha \to \mathcal{P}(\mathcal{P}_\kappa(\lambda)) - \{\emptyset\}$, then there is an $A \in \mathcal{P}_\kappa(\lambda)$ such that $\forall \beta < \alpha\ \exists B \in f(\beta)\ [B \subseteq A]$.

Definition 7. (Weak Square) $\boxminus_{\kappa,\lambda}$: Suppose $f\colon \kappa \to \lambda$ is such that $\forall \alpha < \kappa\ \mathrm{cof}(f(\alpha)) \leq \kappa$. Then there is an $A \subseteq \lambda$, $|A| \leq \kappa$, such that $\forall \alpha < \kappa\ [A \cap f(\alpha)$ is cofinal in $f(\alpha)]$.

Becker showed that if κ is regular and Suslin or the successor of a Suslin, then for all λ within scales

$$\kappa \text{ is } \lambda\text{-supercompact} \leftrightarrow \kappa \text{ is } \lambda\text{-regular}.$$

Using $\boxminus_{\kappa,\lambda}$ for Suslin κ and $\lambda < \Theta$, we show any κ which is regular and Suslin or the successor of a Suslin is λ-regular for all λ within scales.

Using the supercompactness measure on $\mathcal{P}_{\omega_1}(\kappa)$ we establish $\boxminus_{\kappa,\lambda}$ for all Suslin κ and all $\lambda < \Theta$.

This shows an interplay between the type-1 and type-2 theories.

Example: Using weak square we can analyze uniform cofinalities. Consider possible uniform cofinalities of functions $f\colon \omega_2 \to \lambda < \Theta$ with respect to the ω_1-cofinal normal measure μ on ω_2. We have:

Theorem 10. *Suppose $f\colon \omega_2 \to \lambda < \Theta$. Then one of the following holds:*

1. *There is a regular cardinal ρ so that on a μ measure one set f has uniform cofinality ρ.*
2. *On a μ measure one set, $f(\alpha)$ has uniform cofinality α.*
3. *On a μ measure one set, $f(\alpha)$ has uniform cofinality $j_{W_1^1}(\alpha)$.*

Remark 1. $\kappa_1 = \boldsymbol{\delta}^1_4$ is $\boldsymbol{\delta}^2_1$-supercompact, but κ_2, κ_3 are not $\lambda_5 = (\boldsymbol{\delta}^1_5)^-$-supercompact. However $\boxminus_{\kappa,\lambda}$ appears to hold for $\kappa = \kappa_2$, κ_3 and all $\lambda < \Theta$.

There are basic questions open regarding the type-2 theory down low in the $L(\mathbb{R})$ hierarchy, even among the ω_n.

Let ν_n be the supercomapctness measure on $\mathcal{P}_{\omega_1}(\omega_n)$. Let $\kappa(n,m) = j_{\nu_n}(\omega_m)$.

Using order-type result mentioned earlier can show $k(n,m) \geq \omega_{n+2m-1}$. Also, can show equality holds when $n = 2$.

<u>Question</u>: Does equality hold for all n?

Two Applications

Theorem 11. *($ZFC + AD^{L(\mathbb{R})}$) Every projective set is $\aleph_\omega$-Borel.*

Second application uses the extension of the projective hierarchy analysis through the countable levels.

Theorem 12. *(ZFC + $\exists\omega$ Woodin cardinals + a measurable) $\exists\kappa < (\aleph_{\omega_2})^{L(\mathbb{R})}$ [κ is regular in $L(\mathbb{R}) \wedge \kappa$ is not a cardinal in V].*

Future Directions

First, a "negative" result. Let δ be a regular cardinal with the strong partition relation. If $S \subseteq \delta$ is a thin stationary set (i.e., $\forall \alpha \in S$ $[S \cap \alpha$ is not stationary in $S]$), then the c.u.b. filter restricted to S defines a normal measure μ_S on δ.

If S_1, S_2 are thin, define $S_1 \prec S_2$ iff there is a c.u.b. C such that for all $\alpha \in C \cap S_2$, S_1 is stationary in α. This is a wellordering of equivalence classes, where $S_1 \sim S_2$ iff there is a c.u.b. C such that $S_1 \cap C = S_2 \cap C$. Let $o(\delta)$ be the rank of this wellordering.

Theorem 13. *Let δ be a Suslin cardinal with the δ-Suslin sets closed under real quantification. If $\beta < o(\delta)$, then $j_\beta(\delta) < o(\delta)$, where j_β is the embedding corresponding to S_β (the β^{th} stationary set).*

On the positive side:

1) A version of Martin's theorem is true generally.

Theorem 14. *$(AD + V = L(\mathbb{R}))$ Let $\kappa < \boldsymbol{\delta}^2_1$ be a regular limit Suslin cardinal. Let $\lambda < \kappa$ be regular. Let $\boldsymbol{\Gamma}$ be the pointclass closed under $\forall^{\omega^\omega}$ with κ-Suslin$=\exists^{\omega^\omega}\boldsymbol{\Gamma}$. Assume there is a $\boldsymbol{\Delta}$ coding of λ admitting a homogeneous tree U satisfying $(\star)$. Then there is a $\boldsymbol{\Gamma}$-norm $z \to |z|$ on a $\boldsymbol{\Gamma}$-complete set P and a tree T with $P = p[T]$ satifying: the* *is a c.u.b.* $C \subseteq \kappa$ *such that for all* $\alpha \in C$ *of cofinality* λ *there is a* $z \in P$ *with* $|z| = \alpha$ *and* $T_z \restriction \sup_\nu j_\nu(\alpha)$ *illfounded. Here the* ν *range over the homogeneity measures for* U*.*

This gives a start at regular limit Suslin cardinals.

note: This theorem only handles normal measures corresponding to fixed cofinalities.

2) $\boxminus_{\kappa,\lambda}$ is true (assuming $\mathrm{AD} + V = L(\mathbb{R})$) for all Suslin κ and all $\lambda < \Theta$. This should give a method of proceeding at singular Suslin cardinals.

3) New proof of strong partition relation on ω_1 may suggest a way of "decoupling" the analysis of a projective-like hierarchy from the complete inductive analysis below.

Test question: Does every regular Suslin cardinal have the strong partition property?