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Math 136, Second Midterm

1. Define carefully:

(a) A set A is *productive* iff .. there is a total, recursive function f such that for all x

$$W_x \subseteq A \Rightarrow f(x) \in A - W_x$$

(b) A set A is *creative* iff ... A is r.e., and $\overline{A}$ is productive

(c) $A \leq_m B$ iff ... there is a total recursive function f such that for all n

$$n \in A \Leftrightarrow f(n) \in B$$

(d) A is *m-complete r.e.* (or Σ_1 complete) iff ...

A is r.e., and for all r.e. B, $B \leq_m A$.

2.

(a) Show that every productive set has an infinite r.e. subset. Let f witness that A is productive. By the Index Function Thm, we get a total rec. g s.t. $\forall n$

$$W_{g(n)} = W_n \cup \{f(n)\}.$$

Pick e_0 s.t. $W_{e_0} = \emptyset$, and let

$$e_{n+1} = g(e_n).$$

Now let

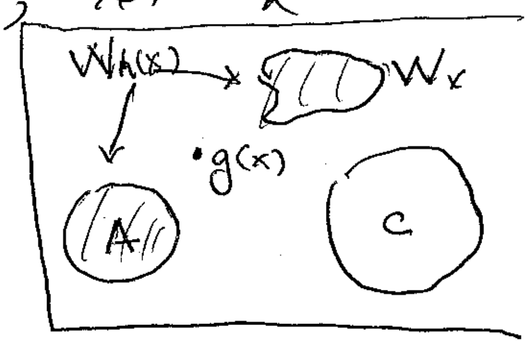
$$x \in B \text{ iff } \exists n \, (x \in W_{e_n}).$$

It is easy to see B is infinite, r.e., and $B \subseteq A$.

(b) Let A and C be disjoint r.e. sets, and suppose that C is creative. Show that $A \cup C$ is creative.

Clearly $A \cup C$ is r.e. Let f witness that $\overline{C}$ is productive. By IFT, let h be total rec. s.t. $\forall x$

$$W_{h(x)} = W_x \cup A.$$



Set

$$g(x) = f(h(x)).$$

Then

$$W_x \subseteq \overline{A \cup C} \Rightarrow W_{h(x)} \subseteq \overline{C} \Rightarrow g(x) \in \overline{C} - W_{h(x)}$$

$$\Rightarrow g(x) \in \overline{A \cup C} - W_x.$$

3. Show there is no partial recursive function ψ such that for all e, if W_e is finite, then $\psi(e)\downarrow$ and W_e has $\leq \psi(e)$ elements. [Hint: one way to go is: given ψ, get an e which "foils ψ" using the recursion theorem.] Suppose ψ were such a function.

By the Index Function Theorem, we get a total recursive function f such that $\forall e$

$$\varphi^{(1)}_{f(e)}(x) = \begin{cases} 0 & \text{if } \psi(e)\downarrow \text{ and } x \leq \psi(e)+1 \\ \uparrow & \text{otherwise.} \end{cases}$$

Thus

$$W_{f(e)} = \begin{cases} \{0, \dots, \psi(e)+1\} & \text{if } \psi(e)\downarrow \\ \emptyset & \text{if } \psi(e)\uparrow \end{cases}$$

Now the recursion theorem gives an e such that

$$W_e = W_{f(e)} .$$

Looking at the defn. of $W_{f(e)}$, we see that it is finite. But then $\psi(e)\downarrow$, and

$$W_e = W_{f(e)} = \{0, \dots, \psi(e)+1\} .$$

So W_e has $\psi(e)+1$ elements, contradiction.

4. Let $f\colon N \to N$ be total, recursive, and one-one. Suppose $\operatorname{ran}(f)$ is not recursive. Put $D = \{i \mid \exists k > i(f(k) < f(i))\}$. Show that $\bar{D}$ has no infinite r.e. subset.

Let $A \subseteq \overline{D}$ be infinite and r.e.

Note

(*) $n \in A \Rightarrow \forall k > n\ (f(k) > f(n))$.

We use this to show $\operatorname{ran}(f)$ is recursive. Clearly $\operatorname{ran}(f)$ is r.e., so it is enough to see $\overline{\operatorname{ran}(f)}$ is r.e. But

(**) $x \notin \operatorname{ran}(f)$ iff $\exists n\ (n \in A \wedge x < f(n) \wedge \forall k < n\ (x \neq f(k)))$,

so $\overline{\operatorname{ran}(f)}$ is r.e.

[To check (**):

$\Rightarrow$ if $x \notin \operatorname{ran}(f)$, then pick any $n \in A$ such that $x < f(n)$. Such n exists as A is infinite and f is 1-1. Then $\forall k < n\ (x \neq f(k))$, so the r.h.s. holds.

$\Leftarrow$ if n witnesses the right hand side of (**), then as $n \in A$, we have $\forall k > n\ x \neq f(k)$ by (*). Thus $x \notin \operatorname{ran}(f)$.]

5. Let Tot $= \{e \mid \phi_e^{(1)}$ is total $\}$. Show that $\bar{K} \leq_m$ Tot.

By the index fcn. theorem, we have a total recursive f such that $\forall e$

$$\varphi_{f(e)}(x) = \begin{cases} 0 & \text{if "}\varphi_e(e)\downarrow \text{ in } x \text{ steps" is not true} \\ \uparrow & \text{otherwise} \end{cases}$$

(Note here that the relation

$R(e,x)$ iff $\varphi_e(e)\downarrow$ in x steps

is decidable.) But then

$$e \in \overline{K} \Rightarrow \varphi_e(e)\uparrow \Rightarrow \forall x\ \varphi_{f(e)}(x)\downarrow \Rightarrow f(e) \in \text{Tot}$$

and

$$e \in K \Rightarrow \exists x\,(\varphi_e(e)\downarrow \text{ in } x \text{ steps}) \Rightarrow \exists x\ \varphi_{f(e)}(x)\uparrow$$

$$\Rightarrow f(e) \in \overline{\text{Tot}}.$$

So $\overline{K} \leq_m$ Tot via f.

6. Let T be a complete, consistent, axiomatizable theory in $\mathcal{L}^{\text{ENT}}$. Show that T is decidable.

Let $T = Thm_\Gamma$, where Γ is decidable.

To decide whether $n = \ulcorner \alpha \urcorner$ for some $\alpha \in T$:

(a) decide whether $n = \ulcorner \alpha \urcorner$ for some sentence α of $\mathcal{L}^{\text{ENT}}$. If not, say "no". If yes, then

(b) simultaneously search for a proof ~~~~ from Γ of α, and for a proof from Γ of $\neg\alpha$.

Since T is complete ~~and consistent~~ you'll find such a proof. If it is a proof of α, say "yes".

If it is a proof of $\neg\alpha$, say "no". This is justified because T is consistent, so that if ~~~~ $\neg\alpha \in Thm_\Gamma$, then $\alpha \notin Thm_\Gamma$.

7. We showed in class that there is a finite set A_E of sentences in $\mathcal{L}^{\text{ENT}}$ such that Thm_{A_E} is m-complete r.e. Use this to show that Val is not decidable, where Val is the set of Gödel numbers of valid sentences in $\mathcal{L}^{\text{ENT}}$.

It is enough to show $\text{Thm}_{A_E} \leq_m Val$.

For this, let

$$\bigwedge A_E = \text{conjunction of axioms of } A_E.$$

Then for any sentence α

$$A_E \vdash \alpha \quad \text{iff} \quad \emptyset \vdash \bigwedge A_E \rightarrow \alpha$$

i.e.

$$\alpha \in \text{Thm}_{A_E} \quad \text{iff} \quad (\bigwedge A_E \rightarrow \alpha) \in Val.$$

So $\text{Thm}_{A_E} \leq_m Val$ via f, where

$$f(n) = \begin{cases} \ulcorner \bigwedge A_E \rightarrow \alpha \urcorner & \text{if } n = \ulcorner \alpha \urcorner \text{ for some sentence } \alpha \text{ of } \mathcal{L}^{\text{ENT}} \\ 0 & \text{otherwise.} \end{cases}$$