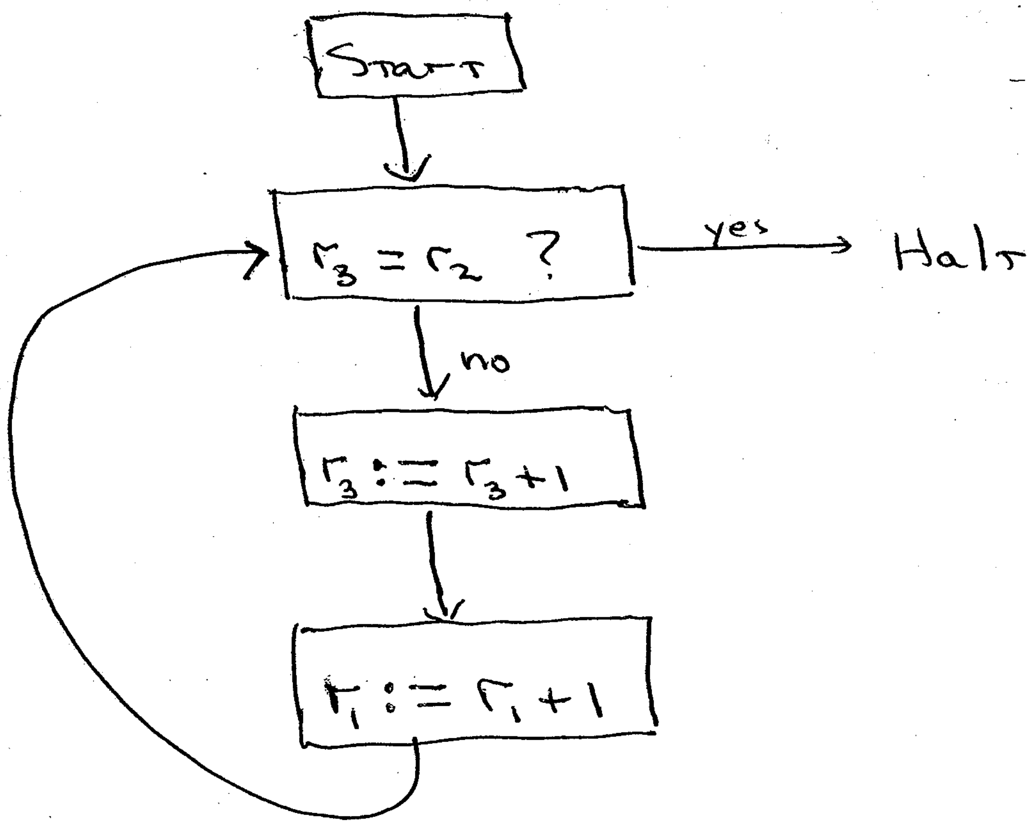
Name: Alan Turing

Signature: Alan Turing

Math 136, First Midterm

1.

(a) Give a flowchart for a URM P which computes the function $f(x, y) = x + y$.



(b) Give the sequence of instructions which is your P from part (a).

$\langle J(2,3,5), S(3), S(1), J(1,1,1) \rangle$

2.

(a) Show that the function $f(x, n) = x^n$ is primitive recursive. (You may assume that addition is a primitive recursive function.)

Multiplication is primitive recursive, since

$$x \cdot 0 = 0$$
$$x \cdot (n+1) = x \cdot n + x$$

(So multiplication is defined by primitive recursion from addition and the constantly 0 function.)

But then $\quad x^0 = 1$

$$x^{n+1} = x^n \cdot x$$

so exponentiation is primitive recursive.

(b) Let $P = \langle T(1, 2), J(3, 4, 1)\rangle$. Recall that $\text{Next}^P(a, b)$ holds iff a and b code computation stages, and a codes a non-halting stage for P, and b codes the next stage dictated by P. Show Next^P is primitive recursive.

$$\text{Next}^P(a,b) \text{ iff } Seq(a) \wedge Seq(b) \wedge lh(a)=2 \wedge lh(b)=2 \wedge Seq((a)_2) \wedge Seq((b)_2) \wedge$$

$$[(a)_1 = 1 \to ((b)_1 = 2 \wedge ((b)_2)_2 = ((a)_2)_1 \wedge \forall j \le lh((b)_2)(j \neq 2 \to ((b)_2)_j = ((a)_2)_j))]$$

and

$$[(a)_1 = 2 \to ((b)_2 = (a)_2 \wedge (((a)_2)_3 = ((a)_2)_4 \to (b)_1 = 1) \wedge (((a)_2)_3 \neq ((a)_2)_4 \to (b)_1 = (a)_1 + 1))]$$

3. (Use back of page if necessary.)

(a) Define carefully: $\phi_e^{(n)}$.

(b) State carefully the Enumeration Theorem.

(c) Let $K(i)$ iff $i \in W_i^{(1)}$. Show K is not decidable.

(d) Explain how the significance of the result in (c) depends on Church's thesis.

(a) If e codes a URM P, then $\phi_e^{(n)} = f_P^{(n)} =$ the ~~fn~~ n-ary function computed by P. Otherwise $\phi_e^{(n)}$ is the empty function.

(b) For any n, the function
$$\Phi_u^{(n)}(e, x_1 \ldots x_n) = \phi_e^{(n)}(x_1 \ldots x_n)$$
is partial recursive.

(c) Suppose K were decidable. Set
$$f(i) = \begin{cases} \Phi^{(2)}(i,i)+1 & \text{if } K(i) \\ 0 & \text{if } \neg K(i) \end{cases}$$

Then f would be total and recursive. So we can fix e s.t. $f = \phi_e^{(1)}$. But then $f(e) = \phi_e^{(1)}(e)$, but $f(e) = \phi_e^{(1)}(e)+1$ by definition. Contradiction!

(d) In part (c), we showed that K is not URM-decidable. (Equivalently, K is not a recursive predicate.) What we would like to know is that K is not decidable by any algorithm, whatsoever. This follows using Church's Thesis.

4. Let f be a partial recursive 1-ary function.

(a) Give an informal description of an algorithm which computes a partial recursive g such that $\forall y \in \mathrm{ran}(f)(f(g(y)) = y)$.

(b) Then give a more formal proof that there is a μ-recursive such function g.

(a) To compute $g(y)$: simultaneously compute $f(0), f(1), f(2), \ldots$ using a dovetailing (or time-sharing) procedure to interleave the individual computations. As soon as you find an x s.t. $f(x) = y$, halt with x as your output.

(b) One way is to use Kleene's T-predicate. Let $\mathcal{P}$ be a URM computing f. So

$$f(x) = \Theta(\mu z\, T^{\mathcal{P}}(x, z)).$$

$g(y)$ will be computed by searching for a pair $\langle x, z\rangle$ s.t. $T^{\mathcal{P}}(x,z) \wedge \Theta(z) = y$.

That is

$$g(y) = \left(\mu t\left[T^{\mathcal{P}}((t)_1, (t)_2) \wedge \Theta((t)_2) = y\right]\right)_1$$

since the μ-operator is being applied to a decidable (in fact, primitive recursive) predicate, our g is μ-recursive.

5. Let $\{A_n \mid n \in N\}$ be a family of infinite sets of natural numbers, and suppose $A_{n+1} \subseteq A_n$ for all n. Suppose also the predicate

$$R(n, x) \Leftrightarrow x \in A_n$$

is decidable. Show that there is a total, recursive, one-one function f such that

$$\forall n(\mathrm{ran}(f) \setminus A_n \text{ is finite}).$$

We define f by

$f(0) = 0$

$$f(n+1) = \mu z[f(n) < z \wedge \underbrace{z \in A_{n+1}}_{R(n+1,z)}]$$

Since the A's are infinite, and R is decidable, f is total and recursive. It is strictly increasing, hence 1-1.

Finally, $\mathrm{ran}(f) \setminus A_n$ is finite, because if $k \geq n$, then $f(k) \in A_k$, and $A_k \subseteq A_n$. So $f(k) \in A_n$ for all $k \geq n$.

6. Let $A(e)$ iff $W_e^{(1)} = \{0\}$. Show A is not decidable.

Sol'n 1 ~~If~~ Let $\mathcal{C} = \{ f \mid f$ is partial rec. $\wedge$ $\operatorname{dom}(f) = \{0\} \}$

Then $\mathcal{C} \neq \emptyset$, and $\mathcal{C} \neq \mathcal{R}$. So by Rice's thm., $\{ e \mid \phi_e^{(1)} \in \mathcal{C} \}$ is not decidable. But this set is just A.

Sol'n 2 We show $K \leq_m A$. For this, we want a total rec. k s.t. $\forall x$

$$\phi_x(x)\downarrow \text{ iff } W_{k(x)}^{(1)} = \{0\}.$$

$$\text{iff } \operatorname{dom} \phi_{k(x)} = \{0\}$$

But let

$$f(x,y) = \begin{cases} 1 & \text{if } y = 0 \wedge \Phi(x,x)\downarrow \\ \uparrow & \text{otherwise} \end{cases}$$

Then f is partial rec., so we have by Index Function theorem a total rec. k s.t. $\forall x, y$

$$\phi_{k(x)}(y) = f(x,y)$$

Clearly, this k is as desired.

7. Show there is a total recursive function s such that for all i, x:

$$i \in W^{(1)}_{s(x)} \text{ iff } \forall k \leq i (k \in W^{(1)}_x).$$

We want

$$\phi^{(1)}_{s(x)}(i)\downarrow \text{ iff } \forall k \leq i \; \phi^{(1)}_x(k)\downarrow$$

So let

$$f(x,i) = \begin{cases} 1 & \text{if } \cancel{\forall k \leq i \, \phi_x} \\ & \forall k \leq i \; \Phi_u(x,k)\downarrow \\ \uparrow & \text{otherwise} \end{cases}$$

It is easy to see f is partial recursive. (To compute $f(x,i)$, just compute $\Phi_u(x,0), \Phi_u(x,1) \cdots \Phi_u(x,i)$ in succession, until all have halted, then give output 1.)

By Index Function, we have s total rec. s.t. $\forall x, i$

$$\phi^{(1)}_{s(x)}(i) = f(x,i)$$

Clearly $\phi_{s(x)}(i)\downarrow$ iff $f(x,i)\downarrow$

iff $\forall k \leq i \; \phi_x(k)\downarrow$

as desired.