Math 136, Incompleteness and Undecidability.

October 27, 2008

The required text for the course is *Computability, an introduction to recursive function theory*, by Nigel Cutland (Cambridge University Press (1980), last reprinted 1997). This will be supplemented at the end of the course by notes prepared by the instructor, based on *A mathematical introduction to logic*, by Herbert Enderton (Harcourt/Academic Press, 2001 edition).

The plan of the course is:

Week 1. Course overview. Unlimited register machines, URM-computable functions. (Cutland, 1.1–1.3.)

Week 2. Closure of URM-computable functions under composition and primitive recursion. Decidable predicates. Primitive recursive functions and predicates. (Cutland 1.4–2.4, 3.3.)

Week 3. Closure of URM-computable functions under a search operator. μ-recursive functions. Gödel numbering computations by a fixed URM in a primitive recursive way. URM-computable = μ-recursive. Church's thesis. (Cutland 2.5, 3.1, 3.2, 3.7. Just mention briefly Turing machines (3.4), and omit 3.5 entirely.)

Weeks 4,5. Countable and uncountable sets. The set of finite sequences from $\mathbb{N}$ is countable, but the set of infinite sequences is not. Gödel numbering URMs, and Kleene's enumeration theorem. Unsolvability of halting problem. $S-m-n$ theorem, many-one reduction, and Rice's theorem. (Cutland, chapters 4 and 5, and 6.1.)

Week 6. Review, leeway, and first midterm.

Week 7. Recursive and recursively enumerable sets. Inseparable r.e. sets. The Rice-Shapiro theorem. (Cutland 6.6, 7.1, 7.2.)

Week 8. Simple, hypersimple, productive, and creative sets. (Cutland 7.3, 7.4.)

Week 9. The Recursion Theorem. Myhill's theorem that all creative sets are many-one equivalent. (Cutland, chapter 11.)

Week 10. Some unsolvable problems: the halting problem for Turing machines, the word problem for semi-Thue systems, context-free grammars. (Cutland 3.4, and instructor's notes drawn from Martin Davis' article in the *Handbook of mathematical logic.*)

Week 11. The structure $\mathcal{N}$ of arithmetic, and its language. Gödel's first incompleteness theorem. (Cutland, chapter 8, but add exponentiation to the basic language, so that one can at least sketch a proof of lemma 1.2.) If time permits, add decidability of the theories of $(\mathbb{Q}, <)$ and of $(\mathbb{N}, +)$.

Week 12. Review, leeway, and second midterm.

Week 13. Define: first-order languages, structures, truth in a structure, validity, and provability. State the Gödel completeness theorem, and its corollary that validity is semi-decidable. If time permits, give a direct proof that if L has only relation symbols, and no $=$ symbol, then the set of validities of L is semi-decidable. (Instructor's notes, based on Enderton 2.2, 2.4, 2.5.)

Week 14. Robinson's arithmetic, in Enderton's version. All Σ_1 truths about $\mathcal{N}$ are provable in Robinson arithmetic. Church's theorem: validity in the language of $\mathcal{N}$ is not decidable. (Instructor's notes, based on Enderton 3.3; see also Cutland 6.5.)

Week 15. More on decidable and undecidable theories: no theory which is true in $(\mathbb{N}, +, \cdot)$, or in $(\mathbb{Z}, +., \cdot)$, or in $(\mathbb{Q}, +, \cdot)$ is decidable; examples. Validity for Σ_1 sentences is decidable, validity for Σ_2 sentences is not. Sturm's algorithm (Cutland 6.4), and Tarski's theorem that the theory of $(\mathbb{R}, +, \cdot)$ is decidable. Hilbert's 10th problem. (Instructor's notes.)

As an alternative, week 15 could give the main ideas in the proof of Gödel's second incompleteness theorem, using notes based on Enderton 3.7.

Grading Policy: Two midterms, worth 20 percent each. Final worth 40 percent. Homework worth 20 percent.

Catalog description:

136. Incompleteness and Undecidability. (4) Three hours of lecture per week. *Prerequisites: 53, 54, and 55.* Functions computable by algorithm, register machines, Turing machines, μ-recursion, Church's thesis. The unsolvability of the halting problem, Rice's theorem. Recursively enumerable sets, the Rice-Shapiro theorem. Simple and creative sets, the Recursion Theorem, Myhill's theorem. Gödel's first incompleteness theorem. Gödel's theorem that validity is semi-decidable, Church's theorem that is is not decidable. Other decidable and undecidable theories. As time permits: Gödel's second incompleteness theorem.(F,Sp)