

Introduction, Course on Geometric Group Theory

The **subject matter** of the course is group theory. Not so much finite groups as infinite groups. “Geometric group theory” was discussed, in a way, in Klein’s “Erlanger Program” of 1872, where the group preserving a geometric structure on some space is discovered to be closely related to the geometry itself. The business of connecting one mathematical subject to another one in Klein’s way was a precursor to the theory of “categories”. Klein’s work contains hints of this, but it was Eilenberg and Mac Lane who, in the early 1940’s brought category theory to the surface and defined functors and natural transformations by means of this theory. Many things in mathematics before the revelation of category theory seemed very complex and detailed and subtle, and are now easy and obvious.

Groups have been seen to be a good way to understand subjects. In topology, for instance, there was the “fundamental group” of a space as defined by Poincaré. But Poincaré did not quite define homology group; he made two attempts, and got it right the second time, and yet he utilized difficult matrix ideas and defined Betti numbers and what we now would call the torsion coefficients. Only decades later did Emmy Noether and L. Vietoris point out that these are numbers that can be used to describe an abelian group; perhaps this was the birth of the notion of **homology group**.

And so, this course will start with a discussion of category theory. In this discussion, many examples will show up, and many definitions will be made. In the printed text, a **definition** of a thing will be denoted by putting the words defined in bold type, so that it will be easy to find them. In the discussion of category theory, many things, groups, free groups, topological spaces, rings, etc., will be defined. Later on, some of these will be given alternative characterizations, so that there will be conceptual repetition. The study of categories soon leads to a need for logical distinctions such as Russell (and Whitehead) called the theory of “Types”; the deep questions of set theory and mathematical “existence” which this leads us to ponder can lead to beautiful dreams.

The majority of topics considered here will be “pure” mathematics, in the sense that there are objects and techniques that lead to abstract facts. It is the case, however, that much of group theory has a computational and algorithmic point of view, and perhaps that is where the study of this subject will lead.

References

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