§ 1. Category Language

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1.0. Introduction

Category theory helps unify the algebraic and topological aspects of mathematics. For example, start with sets with some sort of structure and maps that behave in some way with respect to this structure; in terms of these alone, one can describe notions such as "products". A further level of abstraction leads to maps between categories; the name for such is a "functor". One more step of abstraction leads to maps between functors, and these are called "natural transformations". A very important concept then occurs, the notion of a pair of "adjoint functors". This shows the metaphysical connections between diverse concepts, such as "free group" and "Čech compactification"; try this out on other situations, to see what happens.

This chapter of these notes is not a description of all the delicacies of category theory; you can find that in various books and study it in detail if you feel the urge to do so. (Among good ancient books on the subject are *Abelian Categories* (1964, Harper & Row) by P. Freyd and *Categories for the Working Mathematician* (1971, second edition 1998, Springer GTM 5) by S. Mac Lane.) A more recent book, containing ideas about logic and its relation to category theory is *Categories, Allegories* (1990, Elsevier) by P. Freyd and A. Scedrov.

What this chapter intends to do is to define the elementary aspects of the subject, and to give many kinds of examples, especially in the exercises.

1.1. Definition of "category"

category, object, map, source, target, morphism, homomorphism, continuous, equivariant, subcategory, full subcategory

A **category** $\mathcal{C}$ is a 5-tuple (O, M, s, t, c), where O and M are sets with $O \subset M$, and $s: M \to O$ and $t: M \to O$ are functions. By means of this, define a set $D \subset M \times M$: $D = \{ (\alpha, \beta) \in M \times M : t(\alpha) = s(\beta) \}$. Then, $c: D \to M$ is a function. These are to satisfy the rules:

$\forall\ \alpha, \beta, \gamma \in M$,

(1) $s(c(\alpha, \beta)) = s(\alpha)$.

(2) $t(c(\alpha, \beta)) = t(\beta)$.

(3) If $t(\alpha) = s(\beta)$ and $t(\beta) = s(\gamma)$, then $c(c(\alpha, \beta), \gamma) = c(\alpha, c(\beta, \gamma))$.

(4) $\forall\ \alpha \in O$, $s(\alpha) = t(\alpha) = \alpha$.

(5) $\forall\ \alpha \in M$, $c(s(\alpha), \alpha) = c(\alpha, t(\alpha)) = \alpha$.

Now, discussion and examples give some intuitive understanding: The elements of M are called $\mathcal{C}$-**map**s. The elements of O are $\mathcal{C}$-**object**s or **identity map**s. $s(\alpha)$ and $t(\alpha)$ are the **source** and **target**, respectively, of α. The set D describes the set of pairs of maps whose **composition** $c(\alpha, \beta)$ is defined. A picture of the situation where $S = s(\alpha)$ and $T = t(\alpha)$ is given by the following:

$$S \xrightarrow{\alpha} T$$

or

$$\alpha: S \to T.$$

If $\alpha: S \to T$ and $\beta: T \to U$, then the notation is

$$\beta\alpha: S \to U$$

and think of $\beta\alpha$ as meaning $c(\alpha, \beta)$. Thus, the **arrow** notation is consistent with rules (1) and (2). Furthermore, given $\alpha: A \to B$, $\beta: B \to C$, $\gamma: C \to D$, rule (3) says the associative law $\gamma(\beta\alpha) = (\gamma\beta)\alpha$. It is convenient to consider the $\mathcal{C}$-objects to be part of the $\mathcal{C}$-maps. Given α with source S and target T, rule (5) above says

$$\alpha S = T\alpha = T\alpha S = \alpha.$$

In terminology, a $\mathcal{C}$-map is sometimes called a "**morphism**"; and specific examples have idiosyncratic names: **function** (set-map), **continuous function** (topology-map), **homomorphism** (group-map, ring-map, Λ-module-map), **equivariant mapping** (G-set-map).

If $\mathcal{C}$ is a category, there is an obvious notion of **subcategory** $\mathcal{C}'$; namely, the five pieces of $\mathcal{C}'$ are subsets or restrictions of the five pieces of $\mathcal{C}$ which relate to each other in the obvious way; for example, $O' = s(M') = t(M')$. A particularly useful sort of subcategory is the notion of **full subcategory**: Given a category $\mathcal{C}$ and a subset $O' \subset O$, the full subcategory of $\mathcal{C}$ determined by O', has for its objects O' and for its maps

$$M' = \{ \alpha \in M : s(\alpha) \in O' \text{ and } t(\alpha) \in O' \}.$$

1.2. Extreme cases

monoid, preorder

A category $\mathcal{C}$ having exactly one object is nothing more nor less than a set with an associative binary operation defined on it, for which the single object is the identity. In other words, in this case, it is a **monoid**.

In any category $\mathcal{C}$, one can define a binary relation $\preceq$ on the objects of $\mathcal{C}$, by saying that $A \preceq B$ means that there exists a $\mathcal{C}$-map $A \to B$. This relation is transitive and reflexive. If the category $\mathcal{C}$ has the property that for each pair of objects A and B there is at most one map $A \to B$, then this relation completely determines the category; and conversely, a transitive reflexive relation $\preceq$ given on a set O, describes such a category. This sort of category is called a **preorder**.

1.3. Logic and abstraction

set, Set, category, Category, CCategory, small category, metacategory

In a sense, a category is some kind of algebraic object. It is useful to speak of "categories" which are not sets, however. The simplest example is the thing whose objects are sets, and whose maps are functions. In this case, it is logically strained and circular to call the class of objects a "set", lest some clever individual find a way to derive a contradiction, such as the set of all those sets which are not elements of themselves (Russell's Paradox). This higher degree of abstraction therefore deserves a different name; they are "Categories" with an upper-case "C". Similarly, there are sets and Sets, diagrams and Diagrams, products and Products, etc.

When, inevitably, an even higher level of abstraction is useful, one duplicates the "C". For example, consider all Categories and all structure-preserving maps between them ("functors"); this forms a "**CCategory**". Sometimes it is not clear how certain levels of abstraction should be compared, in which case one simply has to make a guess about how many "C"s should be used.

The more classical terminology is to say **small category** for "category", category for "Category", and **metacategory** for "CCategory". Classically, also, in a Category $\mathcal{C}$, one places the restriction that for each pair of objects a, b of $\mathcal{C}$, the collection $\mathcal{C}[a, b]$ of maps from a to b is a set, with lower-case "s". It is entirely conceivable that by next week someone will have a use for C^3ategories or even for $\mathrm{C}^{\omega^\omega}$ategories.

1.4. Examples

relations, sets, functions, topological space, open set, continuous, Hausdorff, compact Hausdorff, subspace, k-space, monoid, group, abelian group, perfect group, commutator, commutator subgroup, ring, comring, G-set, module

Here are some standard and non-standard examples of categories. They will be useful in getting some understanding, and in providing settings for various exercises.

A. Relations

In this Category, a map consists of a triple (A, B, X), where A and B are sets, and where $X \subset A \times B$. Given a set A, the diagonal is defined to be

$$\Delta_A = \{ (x, y) \in A \times A : x = y \}$$

and we agree to call (A, A, Δ_A) simply "A". Given such a triple $R = (A, B, X)$, define $s(R) = A$ and $t(R) = B$. Given $\alpha: A \to B$ and $\beta: B \to C$, define

$$\beta\alpha = (A, C, \{ (x, z) \in A \times C : \exists\ y \in B \ni (x, y) \in \alpha \text{ and } (y, z) \in \beta \}).$$

In other words, this Category describes binary relations and their relational compositions.

Since the notation $\alpha: A \to B$ makes clear what the first two terms in the triple describing a map are, we agree to use the somewhat ambiguous notation which gives to the third term the name of the entire triple. This allows one to say "$(a, b) \in \alpha$".

B. Sets

Consider those relation-maps $\alpha: A \to B$ such that

$$\forall\ a \in A,\ \ \exists!\ b \in B \ni (a, b) \in \alpha.$$

In other words, this is the "subCategory" of relations which have the peculiar asymmetric property ascribed to "well-defined, single-valued" **functions of sets**.

C. Topology

A **topological space** consists of a pair $(A, \mathcal{U})$, where A is a set, and $\mathcal{U}$ is a set of subsets of A which is closed under finite intersections and arbitrary unions, such that both A and the empty set $\emptyset$ are elements of $\mathcal{U}$. The elements of $\mathcal{U}$ are called **open set**s of $(A, \mathcal{U})$. Given two topological spaces $(A, \mathcal{U})$ and $(A', \mathcal{U}')$, a set-map $f: A \to A'$ is said to be a **continuous** function from the first topological space to the second, whenever: $\forall\ P \in \mathcal{U}'$, $f^{-1}(P) \in \mathcal{U}$. We agree to use the ambiguous terminology that calls the pair $(A, \mathcal{U})$ simply A.

A topology-map is a triple (A, B, f), where A and B are topological spaces and $f: A \to B$ is a continuous function. The topology Category has many interesting full subCategories, which are determined by limiting the class of topology-objects in one way and another.

D. Hausdorff

A Hausdorff space is a topological space A satisfying the following axiom:

$$\forall\ x, y \in A,\ \text{if } x \neq y,\ \text{then } \exists\ P, Q \in \mathcal{U} \ni x \in P \text{ and } y \in Q \text{ and } P \cap Q = \emptyset.$$

[I believe it was Lefschetz who described this axiom as meaning that x and y can be "housed off" from each other.]

E. Compact Hausdorff

A topological space A is said to be compact if every set of open sets whose union is A contains a finite subset whose union is A. Spaces which are both compact and Hausdorff have properties which are especially appealing to analysts, because they have many continuous functions to the real line, and because they have a sort of **finite feeling** about them.

F. k-spaces

Let $(A, \mathcal{U})$ be a topological space, and let $B \subset A$. Define

$$\mathcal{U}_B = \{\, Q \subset B\ :\ \exists\ P \in \mathcal{U} \ni Q = B \cap P\,\}.$$

Then $(B, \mathcal{U}_B)$ is a topological space, called the **subspace** B of A.

Now, a given space A will in general have many compact subspaces. Let A be a Hausdorff space satisfying

$\forall\ P \subset A$, if $\forall\ B \subset A$,

if B is compact, then $B \cap P$ is open in the subspace B,

then P is open in A.

That is, a subset of A is characterized as an open subset by the condition that its intersection with every compact subspace of A is open in that subspace. Call such a space A a k-space. For example, compact Hausdorff spaces are k-spaces, and it is not hard to show that metrizable spaces are k-spaces. k-spaces are particularly useful in the study of function-spaces with the compact-open topology.

G. Monoids

A monoid is a category with exactly one object. More efficiently, a **monoid** can be described as a triple $(A, 1, \mu)$, where $1 \in A$ and $\mu\colon A \times A \to A$ is a binary operation, such that μ is associative and 1 is a two-sided identity. A monoid-map $\varphi\colon (A, 1, \mu) \to (A', 1', \mu')$ consists of a set-map $\varphi_1\colon A \to A'$ such that $\varphi_1(1) = 1'$ and $\varphi_1(\mu(a, b)) = \mu'(\varphi_1(a), \varphi_1(b))$. This describes the Category of monoids.

We agree to use the shorthand notation A for $(A, 1, \mu)$ and φ for φ_1. There are two customary sorts of notation: "Multiplicative notation" uses 1 for 1, and ab or $a \cdot b$ for $\mu(a, b)$; "additive notation" uses 0 for 1 and $a + b$ for $\mu(a, b)$. In general, additive notation is for **commutative** monoids, in which $a + b = b + a$.

H. Groups

A group is a monoid G in which

$$\forall\ g \in G,\ \exists\ h \in G \ni gh = hg = 1.$$

Given $g \in G$, the element h satisfying this axiom is unique; it is the "**inverse** of g" and denoted by g^{-1} or $\overline{g}$, or in additive notation $-g$.

The Category of groups is the corresponding full subCategory of the Category of monoids.

It is a fact that follows from this description that a group-map $\varphi\colon G \to G'$ satisfies the rule $\varphi(g^{-1}) = (\varphi(g))^{-1}$.

I. Abelian groups

An abelian group is a group in which the binary operation is commutative. This then determines, as a full subCategory of the Category of groups, the Category of abelian groups.

J. Perfect groups

Let G be a group, with $a, b \in G$. The **commutator** $[a, b]$ is defined to be $[a, b] = aba^{-1}b^{-1}$. For $n \geq 0$, and $a_1, \dots, a_n \in G$, define

$$\prod_{i=1}^{n}[a_i, b_i] = P_n$$

recursively: $P_0 = 1$ and $P_n = P_{n-1} \cdot [a_n, b_n]$. Then the set of those elements of G which can be written as $\prod_{i=1}^{n}[a_i, b_i]$ for some $n \geq 0$ and some $a_i, b_i \in G$, is called the **commutator subgroup** of G. If G equals its commutator subgroup, call G a **perfect group**.

The Category of perfect groups (as a full subCategory of the Category of groups) is useful as an interesting example.

K. Rings

A ring is a 5-tuple $(\Lambda, \mu, \pi, 0, 1)$, where μ and π are binary operations on Λ, written, respectively, as $+$ and $\cdot$, where $(\Lambda, 0, \mu)$ is an abelian group, $(\Lambda, 1, \pi)$ is a monoid, and the two distributive laws hold. Thus, in these notes, "ring" always denotes "ring with unity". A ring-map $\alpha\colon \Lambda \to \Lambda'$ is a set-map which is both an abelian group-map and a monoid-map; in particular, a ring-map here always preserves the multiplicative identity 1.

L. Comrings

A **comring** is a ring in which the multiplicative monoid is commutative; *i.e.*, a comring is a "commutative ring with unity". The Category of comrings is the corresponding full subCategory of the Category of rings.

M. G-sets

Let G be a fixed group, using multiplicative notation. A G-set is a pair (X,μ), where X is a set and $\mu: G\times X \to X$ is a set-map satisfying the rules

$$\forall\ x\in X,\ \forall\ a,b\in G,\ \mu(1,x)=x \text{ and } \mu(ab,x)=\mu(a,\mu(b,x)).$$

A map of G-sets, $\beta:(X,\mu)\to(X',\mu')$, is a set-map $\beta_1: X\to X'$ which satisfies the rule

$$\forall\ a\in G,\ \forall\ x\in X,\ \beta_1(\mu(a,x))=\mu'(a,\beta_1(x)).$$

The notation $a\cdot x$ means $\mu(a,x)$. And often one says X for (X,μ) and β for β_1.

N. Λ-modules

Let Λ be a ring. A Λ-**module** is a pair (A,μ), where A is an abelian group, using additive notation, and $\mu:\Lambda\times A\to A$ is a set-map (on the underlying sets). To state the rules which must be satisfied, the notation $\alpha * x$ means $\mu(\alpha,x)$. The rules are: $\forall\ \alpha,\beta\in\Lambda,\ \forall\ x,y\in A$,

$$1*x=x,\ (\alpha\cdot\beta)*x=\alpha*(\beta*x),\ \alpha*(x+y)=(\alpha*x)+(\alpha*y), \text{ and}$$

$$(\alpha+\beta)*x=(\alpha*x)+(\beta*x).$$

Note that in the last equation, the first "+" is the binary operation in the additive group of Λ and the second is the binary operation in the abelian group A. Henceforward, we agree to use the notation "$\alpha\cdot x$" or "αx" in place of "$\alpha * x$".

1.5. Duality

dual category

Given a category $\mathcal{C}$, since there is a symmetry in the rules that a category satisfies, one can interchange source and target and the order of the composition and obtain a category $\mathcal{C}^*$, the **dual category** to $\mathcal{C}$. Specifically, let $\mathcal{C}=(O,M,s,t,c)$, and define $\mathcal{C}^*=(O,M,s^*,t^*,c^*)$, where $s^*=t$, $t^*=s$, and $c^*(\alpha,\beta)=c(\beta,\alpha)$. The set of pairs of maps $D^*=\{\,(\beta,\alpha)\in M\times M\ :\ (\alpha,\beta)\in D\,\}$.

Since it is convenient not to confuse the maps in $\mathcal{C}$ with those in $\mathcal{C}^*$, we agree to add the "*" to the symbols for these maps. Thus, when $\alpha: S\to T$ is a $\mathcal{C}$-map, one denotes the same map as a $\mathcal{C}^*$-map by $\alpha^*: T^*\to S^*$.

Many of the definitions within category theory come in dual pairs. It is convenient and enlightening to define one concept in such a dual pair and to define the other by calling it the dual concept.

1.6. Directed graphs and commutative diagrams

directed graph, edge, vertex, directed path, reduce, diagram, commutative diagram

A **directed graph** Γ is a 4-tuple (V, E, s, t) where $V \subset E$ and s and t are functions $E \to V$ satisfying the rule

$$\forall\ v \in V,\ \ s(v) = t(v) = v.$$

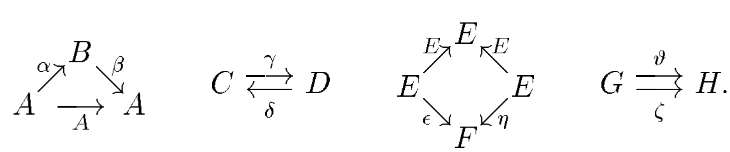
Thus, a directed graph is just a category-like thing without a composition operation. The elements of E are the **edges**, and the elements of V are the **vertices**.

The category $\mathcal{P}(\Gamma)$ of **directed paths** in Γ is defined as follows: A (directed) path in Γ is a finite-tuple $p = (e_1, \ldots, e_n)$ of elements $e_i \in E$, such that, for all $i = 1, \ldots, n-1$, $t(e_i) = s(e_{i+1})$. Such a path **reduces** by crossing out elements e_i which belong to V, except that a path does not reduce to the empty-tuple; each path then reduces to a unique smallest path which contains no vertices unless the length of the path is 1; the maps of $\mathcal{P}(\Gamma)$ are these smallest paths. Say that the path p has source $s(e_1)$ and target $t(e_n)$. If a path p has target equal to the source of a path q, one **concatenates** them to get a path pq; in a few exceptional cases, pq may reduce, so as to get an official $\mathcal{P}(\Gamma)$-map; this is the composition operation in $\mathcal{P}(\Gamma)$.

Suppose $\mathcal{C}$ is some category. A **diagram** in $\mathcal{C}$ modeled on the directed graph Γ, is a function D from the edges of Γ to the maps of $\mathcal{C}$, such that, for each vertex v of Γ, $D(v)$ is an object of $\mathcal{C}$. In other words, a diagram is a directed graph labeled with maps in $\mathcal{C}$. Each directed path in the graph yields, by composing the edge-labels, a $\mathcal{C}$-map.

A diagram is said to be commutative (= consistent) when, for every pair of vertices u and v, and for every pair of paths p and q each having source u and target v, p and q yield equal $\mathcal{C}$-maps.

For example, consider the four diagrams:



To say that the first diagram is commutative is to assert $\beta\alpha = A$; to say that the second diagram is commutative is to assert both $\gamma\delta = D$ and $\delta\gamma = C$. To say that the third diagram is commutative says nothing except that ϵ and η have the same source E and the same target F, while commutativity of the fourth diagram asserts that $\vartheta = \zeta$. (Recall that A is the "identity map" $A \to A$.)

The use of diagrams provides an elegant mathod for depicting in pictures various facts about categories.

1.7. Cancellation in a category

retraction, coretraction, equivalence, monic, epic, terminal object, initial object

If $\alpha: A \to B$ and $\beta: B \to A$ are maps in a category $\mathcal{C}$ and $\beta\alpha = A$, then β is a left inverse of α and α is a right inverse of β. In the world of categories, the terminology is: If α has a left inverse, then α is a **coretraction**; and if β has a right inverse, then β is a **retraction**. These are dual notions.

If α has both a left inverse and a right inverse, then these are equal and one says that α is a $\mathcal{C}$-**equivalence**. Equivalence is a self-dual notion.

If $\alpha: A \to B$ has the property, whenever the compositions are defined and have the same source, that $\alpha x = \alpha y$ implies $x = y$, then α is left-cancellable. In category language, the terminology is that α is **monic**. The word for the dual concept, when α is right-cancellable, is that α is **epic**.

An object I in a category $\mathcal{C}$ is called an **initial object**, when, for every object X, there is a unique map $I \to X$. The dual definition describes what it means for T to be a **terminal object** in $\mathcal{C}$.

1.8. Limits and colimits

cone above, cone below, category above, category below, limit, colimit, product, coproduct, pullback, pushout, equalizer, coequalizer, inversely directed set, inverse limit, directly directed set, direct limit

Let Γ be a directed graph. Adjoin to Γ a new vertex v_0 and, for all vertices w of Γ, new edges $e_w: v_0 \to w$. This produces a new graph $v_0\Gamma$, the **cone above** Γ. The dual construction, with new vertex v_1 and new edges $e'_w: w \to v_1$, is written Γv_1 and called the **cone below** Γ.

Suppose $\mathbf{D}$ is a commutative diagram in the Category $\mathcal{C}$, modeled on Γ. Consider a commutative diagram modeled on $v_0\Gamma$ which extends $\mathbf{D}$; this is called an object above $\mathbf{D}$; it is described, pictorially, by

$$\boldsymbol{\alpha}: A \to \mathbf{D}.$$

Here A is a $\mathcal{C}$-object, and $\boldsymbol{\alpha}$ consists of a collection of $\mathcal{C}$-maps $\{\alpha_w: A \to D_w\}$, where D_w is the $\mathcal{C}$-object associated by $\mathbf{D}$ to the vertex w of Γ.

A map above $\mathbf{D}$ is a commutative diagram of this sort:

$$\begin{array}{ccc} A & \xrightarrow{\varphi} & B \\ & {\scriptstyle\boldsymbol{\alpha}}\searrow \quad \swarrow{\scriptstyle\boldsymbol{\beta}} & \\ & \mathbf{D} & \end{array}$$

Here, $\varphi: A \to B$ is a single $\mathcal{C}$-map. This diagram is modeled on the graph consisting of two cones above Γ with different top vertices v_0 and v_0' and another edge $e: v_0 \to v_0'$.

The Category "above $\mathbf{D}$", consists of objects and maps above $\mathbf{D}$. The dual notion is the Category "below $\mathbf{D}$".

If **D** is a diagram in the Category $\mathcal{C}$, then perhaps the Category above **D** has a terminal object; if so, this terminal object is called the **limit** of **D**. It is unique in that for any two terminal objects in the Category above **D**, there is a unique equivalence in the Category above **D** from one to the other. This terminal object consists of an object A in the Category $\mathcal{C}$ together with a collection of $\mathcal{C}$-maps $\boldsymbol{\alpha}: A \to \mathbf{D}$, making a commutative diagram.

The dual notion, an initial object in the Category below **D**, is called the **colimit** of **D**.

The limit and colimit of **D** are sometimes written $\lim \mathbf{D}$ and $\operatorname{colim} \mathbf{D}$.

Now, there are certain kinds of limits and colimits which have special names.

1.8.1. Product and coproduct

Suppose that the graph Γ on which the diagram **D** is modeled has only vertices and no other edges. In this case, think of **D** as a set of $\mathcal{C}$-objects $\{ A_w : w \in \Gamma \}$ indexed on the vertices w of Γ. Commutativity of the diagram **D** is automatic. The limit of this diagram is called the **product** of $\{ A_w : w \in \Gamma \}$, and it is denoted by the following sorts of symbols:

$$\prod_{w \in \Gamma} A_w \qquad \prod \{ A_w : w \in \Gamma \} \qquad A \times B \qquad A \sqcap B.$$

The last two kinds of notation indicate that Γ consists of just two vertices.

The dual idea is called the **coproduct** of the indexed set of $\mathcal{C}$-objects. There is notation such as this:

$$\coprod_{w \in \Gamma} A_w \qquad \coprod \{ A_w : w \in \Gamma \} \qquad A \amalg B \qquad A \sqcup B.$$

1.8.2. Pullback and pushout

Consider a diagram of the following shape:

$$A \stackrel{\alpha}{\to} B \stackrel{\beta}{\leftarrow} C$$

Every diagram of this sort is automatically commutative. Its limit is called the **pullback** of α and β. A commutative "square" of the following type

$$\begin{array}{ccccc} & \beta' & P & \alpha' & \\ & \swarrow & & \searrow & \\ A & \underset{\alpha}{\to} & B & \underset{\beta}{\leftarrow} & C \end{array}$$

is called a **pullback diagram**, if P, decorated with the maps indicated, and their compositions, is the pullback of the diagram on the bottom row. Of course, we agree simply to call P itself the pullback of the diagram, the maps β' and α' and the commutativity of the resulting diagram being understood tacitly.

The dual concept is that of **pushout** and **pushout diagram**, involving commutative "squares" of the following kind:

$$\begin{array}{ccccc} A & \stackrel{\alpha}{\leftarrow} & B & \stackrel{\beta}{\rightarrow} & C \\ & {\scriptstyle \beta'}\searrow & & \swarrow{\scriptstyle \alpha''} & \\ & & Q & & \end{array}$$

1.8.3. Equalizers and coequalizers

A diagram which is not commutative can be expanded into a commutative diagram by adjoining edges to the graph Γ to which are associated identity maps. A specific example is

$$A \underset{\beta}{\overset{\alpha}{\rightrightarrows}} B$$

which can be expanded to

$$\begin{array}{ccccc} & & A & & \\ & {\scriptstyle A}\swarrow & & \searrow{\scriptstyle \alpha} & \\ A & & & & B. \\ & {\scriptstyle A}\nwarrow & & \nearrow{\scriptstyle \beta} & \\ & & A & & \end{array}$$

The limit of this diagram is called the **equalizer** of (α, β).

The dual concept is the **coequalizer**.

1.8.4. Inverse and direct limits

A set D with a transitive, reflexive relation $\preceq$ on it is a "preorder". (Note that, as in Section 0.2, if $x \preceq y$, the arrow goes from x to y.) It is a category, and so it is a directed graph. If it satisfies the additional rule:

$$\forall\, a, b \in D,\ \exists\, c \in D \ni c \preceq a \text{ and } c \preceq b$$

then call D an **inversely directed** set. A commutative diagram in $\mathcal{C}$ which is modeled on an inversely directed set is called an **inverse system**, and its limit is called the **inverse limit** of the inverse system.

Dually, a preorder satisfying the rule

$$\forall\, a, b \in D,\ \exists\, c \in D \ni a \preceq c \text{ and } b \preceq c$$

is a **directly directed** set. A diagram modeled on it is called a **direct system**, and its colimit is called the **direct limit**.

1.9. Functors

functor, identity functor, preserves limit or colimit, inclusion functor, forgetful functor

Let $\mathcal{A} = (O_1, M_1, s_1, t_1, c_1)$ and $\mathcal{B} = (O_2, M_2, s_2, t_2, c_2)$ be two categories. A **functor** $\Phi \colon \mathcal{A} \to \mathcal{B}$ is a function from M_1 to M_2 which is compatible (in the obvious algebraic sense) with the category-structure. In particular Φ takes O_1 into

O_2, and preserves source, target, and composition. It follows that Φ will take any commutative diagram $\mathbf{D}$ in $\mathcal{A}$ into a commutative diagram $\Phi\mathbf{D}$ in $\mathcal{B}$, both diagrams being modeled on the same graph.

It is clear how to define the "identity functor" from $\mathcal{A}$ to $\mathcal{A}$, and that the composition of a functor $\Phi\colon \mathcal{A} \to \mathcal{B}$ with a functor $\Psi\colon \mathcal{B} \to \mathcal{C}$ is a functor $\Psi\Phi\colon \mathcal{A} \to \mathcal{C}$. Thus, the functors on categories form a Category. And functors on Categories form a CCategory.

Suppose that $\Phi\colon \mathcal{A} \to \mathcal{B}$ is a functor and that $\mathbf{D}$ is a diagram in $\mathcal{A}$. If the limits and colimits exist, then there is a unique map (of diagrams above $\Phi(\mathbf{D})$)

$$\Phi(\lim \mathbf{D}) \to \lim \Phi(\mathbf{D}).$$

If this is an equivalence, one says that Φ preserves the limit of $\mathbf{D}$. It is not unusual for certain functors to preserve all limits. Dually, there is a unique map

$$\operatorname{colim} \Phi(\mathbf{D}) \to \Phi(\operatorname{colim} \mathbf{D}).$$

If that is an equivalence, one says that Φ preserves the colimit.

A functor which is the inclusion of a subCategory into a Category will be called an **inclusion functor**; to designate an inclusion functor, if one comes up, use $\mathbf{I}$ or $\mathbf{J}$. For example, the inclusion $\mathbf{I}$: abelian groups $\to$ groups.

Certain categories are related to others by involving objects with greater structure. For example, a topological space is a set with additional structure, and topology-maps are set-maps of the underlying space; there results a functor

$$\mathbf{F}\colon \text{topology} \to \text{sets}.$$

This sort of functor is called a **forgetful functor**. Other examples are:

$$\mathbf{F}_1\colon \text{rings} \to \text{abelian groups}$$

$$\mathbf{F}_2\colon \text{rings} \to \text{monoids}.$$

1.10. Completeness

infimum, supremum, complete

Suppose, first, that $\mathcal{C}$ is a preorder. Let $\mathbf{D}$ be a subset of the set of objects of $\mathcal{C}$; imagine $\mathbf{D}$ to be a diagram, in which some of the maps of $\mathcal{C}$ between these objects are included. Whatever maps are included, $\mathbf{D}$ is a commutative diagram. The limit of $\mathbf{D}$ is just an **infimum** of the set of objects in $\mathbf{D}$, and thus does not depend on which maps were included. This infimum s has the defining property

$$\forall\, x \in \mathbf{D},\ s \preceq x \text{ and}$$

$$\forall\, y \in \mathcal{C},\ \text{if } \forall\, x \in \mathbf{D},\ y \preceq x,\ \text{then } y \preceq s.$$

Now, suppose that *every* subset $\mathbf{D}$ of $\mathcal{C}$ has an infimum. Then, given any subset $\mathbf{D}$, the set $\mathbf{D}' = \{\, z \in \mathcal{C} : \forall\, x \in \mathbf{D},\ x \preceq z \,\}$ has an infimum, and this is the "**supremum**" of $\mathbf{D}$. In other words, in a preorder in which all limits exist, all colimits exist.

A rather similar argument shows that, in any one of the Categories discussed so far, if all limits of diagrams $\mathbf{D}$, based on graphs Γ which involve only (small) sets, exist, then all such colimits exist. The general statement which is true has to have delicate set-theoretic clauses, because the analogue of $\mathbf{D}'$ above will be a Diagram rather than a diagram, and some way of limiting the Diagram down to set size has to be hypothesized; for these technicalities, see [Freyd, Chapter 3].

Call a Category in which all limits and colimits exist "complete". This extends the notion of a complete lattice.

1.11. Natural transformations

natural, Yoneda's Lemma

Let $\mathcal{A} = (O, M, s, t, c)$ and $\mathcal{B} = (O', M', s', t', c')$ be two categories. Let $\Phi, \Psi : \mathcal{A} \to \mathcal{B}$ be two functors. Consider a function E from the objects O of $\mathcal{A}$ to the maps M' of $\mathcal{B}$, such that, for each $\mathcal{A}$-object A,

$$E(A) : \Phi(A) \to \Psi(A)$$

and such that, for each $\mathcal{A}$-map $\alpha : A \to B$, the following diagram of $\mathcal{B}$-maps is commutative.

$$\begin{array}{ccc} \Phi(A) & \xrightarrow{\Phi(\alpha)} & \Phi(B) \\ {\scriptstyle E(A)}\downarrow & & \downarrow{\scriptstyle E(B)} \\ \Psi(A) & \xrightarrow[\Psi(\alpha)]{} & \Psi(B) \end{array}$$

Such a function E is said to be a **natural** map from Φ to Ψ. This describes the set of maps in a new category, **natural**$(\mathcal{A}, \mathcal{B})$. The objects in this category are the functors $\mathcal{A} \to \mathcal{B}$, with identity natural maps from Φ to Φ, given by $E(A) = \Phi(A)$, and with composition defined by $(E_1 E_2)(A) = (E_1(A))(E_2(A))$.

In order to try to be less confusing, we agree to use the notation E_A for $E(A)$. A natural map is also called a "natural transformation". In case $\mathcal{A}$ and $\mathcal{B}$ are Categories, it may be wise to call **natural**$(\mathcal{A}, \mathcal{B})$ a CCategory.

Recall that a functor from $\mathcal{C}$ to **sets** can be described, sometimes, by representing it by a $\mathcal{C}$-object A; this functor Φ, roughly speaking, is given by the formula $\Phi(B) = \mathcal{C}[A, B]$, and so denote it by $\mathcal{C}[A, -]$. There is a simple way to describe natural maps with source functor $\mathcal{C}[A, -]$.

Yoneda's Lemma. Given a functor $F : \mathcal{C} \to$ **sets** and a $\mathcal{C}$-object A and an element $\varepsilon \in F(A)$, there is a unique natural map $E : \mathcal{C}[A, -] \to F$ such that $\varepsilon = E_A(A)$. The formula for $E_B : \mathcal{C}[A, B] \to F(B)$ is this: Let $\varphi \in \mathcal{C}[A, B]$. Then $E_B(\varphi) = F(\varphi)(\varepsilon)$.

Sketch of proof. The proof is based on an understanding of all the terms involved and looking at the following diagram:

$$\begin{array}{ccccc} A \in & \mathcal{C}[A,A] & \xrightarrow{[A,\varphi]} & \mathcal{C}[A,B] & \ni \varphi \\ & {\scriptstyle E_A}\downarrow & & \downarrow{\scriptstyle E_B} & \\ \varepsilon \in & F(A) & \xrightarrow[F(\varphi)]{} & F(B) & \end{array}$$

This diagram describes what $E_B(\varphi)$ has to be, and the fact that this describes a natural transformation involves chasing diagrams using the associativity of composition in a category. □

The CCategory **natural**$(\mathcal{A},\mathcal{B})$ admits the usual category constructions. Limits and colimits are easy to ascertain. An equivalence in this CCategory is called a **natural equivalence**. A natural equivalence E from Φ to Ψ is characterized by the fact that, for each $\mathcal{A}$-object A, the $\mathcal{B}$-map $E_A\colon \Phi(A) \to \Psi(A)$ is a $\mathcal{B}$-equivalence.

1.12. Adjoint functors

adjoint functors, left and right adjoint, adjoint functor theorem

Let $\mathcal{A}$ and $\mathcal{B}$ be two Categories, $L\colon \mathcal{A} \to \mathcal{B}$ and $R\colon \mathcal{B} \to \mathcal{A}$ two functors. For each pair of objects, A in $\mathcal{A}$ and B in $\mathcal{B}$, consider the set $\mathcal{B}[L(A), B]$ and the set $\mathcal{A}[A, R(B)]$. Each of these describes a functor from $\mathcal{A}^* \times \mathcal{B}$ to **sets**. If there is a natural equivalence between these two functors, one writes

$$\mathcal{B}[L(A), B] \approx \mathcal{A}[A, R(B)]$$

and says that (L, R) is a pair of **adjoint functors**. One calls L the **left adjoint** of R and R the **right adjoint** of L.

Given such a situation, if $\mathbf{D}$ is a commutative $\mathcal{B}$-diagram having a limit $\lim \mathbf{D}$ of the form $\boldsymbol{\alpha}\colon B \to \mathbf{D}$, apply the functor R to get an $\mathcal{A}$-object above $R(\mathbf{D})$, namely $R(\boldsymbol{\alpha})\colon R(B) \to R(\mathbf{D})$. Then, given any $\mathcal{A}$-object A above $R(\mathbf{D})$, involving, say, $A \to R(\mathbf{D})$, apply the above adjointness equivalence; find that there is a unique $\mathcal{B}$-map from $L(A)$ to B in the Category above $\mathbf{D}$, because $B = \lim \mathbf{D}$. Thus, there is just one map in the Category above $R(\mathbf{D})$ from A to $R(B)$. In other words, $R(\lim \mathbf{D}) = \lim R(\mathbf{D})$ and R preserves limits. *Ergo*:

Theorem. *A right adjoint preserves limits. Dually, a left adjoint preserves colimits.*

Now, this theorem has a converse, the **adjoint functor theorem**, that if a functor $R\colon \mathcal{B} \to \mathcal{A}$ preserves limits, if $\mathcal{B}$ is a complete category (thus having all kinds of limits), and if an additional logical condition is true so that certain Limits in the construction can be reduced to limits, then R has a left adjoint. For a careful statement of this theorem, see [Freyd, Chapter 3].

1.13. Exercises and Examples

General Facts.

1. Use the notation $\mathcal{C}[A,B]$ for the set of $\mathcal{C}$-maps from A to B. Then $\mathcal{C}[A,A]$ is a monoid, called the monoid of self-maps of A; and the subset consisting of those self-maps which are equivalences, forms a group $\mathrm{SE}\mathcal{C}[A,A]$, called the group of self-equivalences of A. Self-maps are sometimes called "**endomorphism**s" and self-equivalences are known as "**automorphism**s". In the case of the Category of sets, a self-equivalence is called a **permutation**.

2. Every coretraction is monic and every retraction is epic.

3. If $\beta\alpha$ is monic, then α is monic.

4. Find an example where $\beta\alpha$ is monic, α is epic, and β is not monic.

5. An epic coretraction is an equivalence.

6. If I and J are initial objects in a category $\mathcal{C}$, then there is a unique equivalence $I \to J$.

7. If I is an initial object, then every map $X \to I$ is epic.

8. (To sharpen intuition.) Find an example of an initial object I in some category, and a map $I \to X$ which is not monic.

9. In a pullback diagram such as

$$\begin{array}{ccccc} & {\scriptstyle\beta'} & P & {\scriptstyle\alpha'} & \\ & \swarrow & & \searrow & \\ A & \underset{\alpha}{\to} & B & \underset{\beta}{\leftarrow} & C \end{array}$$

if α is monic, then α' is monic. One says, therefore, that the pullback of a monic map is monic. Dually, the pushout of an epic map is epic.

10. (Intuition sharpener.) Find a Category $\mathcal{C}$ and a pullback diagram in $\mathcal{C}$, as above, such that both α and β are epic, but neither α' nor β' is epic.

11. (Basic stuff.) In any set, the class of finite subsets forms a directly directed set under inclusion $\subset$. In any topological space, the family of compact subspaces forms a directly directed set under inclusion; $k(X)$ is in fact the direct limit of the family of compact subspaces of X. Any group is the direct limit, in the Category of groups, of the family of finitely generated subgroups.

12. The forgetful functors

$$F_1\colon \mathbf{topology} \to \mathbf{sets}$$

$$F_2\colon \mathbf{groups} \to \mathbf{sets}$$

$$F_3\colon \mathbf{rings} \to \mathbf{sets}$$

preserve products and pullbacks, and, in fact, all limits.

13. A commutative diagram is essentially a functor from a preorder into a Category $\mathcal{C}$.

14. Let A be a fixed object in the Category $\mathcal{C}$. For any object B of $\mathcal{C}$, let $F(B) = \mathcal{C}[A, B]$. F can be construed as a functor $F\colon \mathcal{C} \to$ **sets**. The functor is said to be represented by the object A. Such a functor F preserves limits.

15. Functors on (small) categories constitute a Category. As in the case of other algebraic Categories, the product of two categories has a simple description. Let $\mathcal{C} = (O, M, s, t, c)$ and $\mathcal{C}' = (O', M', s', t', c')$. The product category $\mathcal{C} \times \mathcal{C}' = (O'', M'', s'', t'', c'')$ has for objects $O'' = O \times O'$ (the Cartesian product of the sets of objects); for maps $M'' = M \times M'$; for source, $s''(\alpha, \beta) = (s(\alpha), s'(\beta))$; for target, $t''(\alpha, \beta) = (t(\alpha), t'(\beta))$; composition, $c''((\alpha, \beta), (\gamma, \delta)) = (c(\alpha, \gamma), c'(\beta, \gamma))$.

Likewise, there is the **product of two Categories**, in the CCategory of functors of Categories. Given a Category $\mathcal{C}$, there is a functor

$$\Phi\colon \mathcal{C}^* \times \mathcal{C} \to \mathbf{sets}$$

described as follows: If $E^* \times F$ is an object of $\mathcal{C}^* \times \mathcal{C}$, then

$$\Phi(E^* \times F) = \mathcal{C}[E, F]$$

the set of $\mathcal{C}$-maps from E to F. Let $\alpha^*\colon A^* \to B^*$ be a map in $\mathcal{C}^*$, and $\beta\colon C \to D$ be a map in $\mathcal{C}$; then $\alpha\colon B \to A$ is a $\mathcal{C}$-map. Let

$$\varphi \in \mathcal{C}[A, C] = \Phi(A^*, C).$$

The definition is

$$\Phi(\alpha^*, \beta)(\varphi) = \beta\varphi\alpha\colon B \to D.$$

In other words, Φ is a way of discussing maps and their compositions in $\mathcal{C}$ as a functor from $\mathcal{C}^* \times \mathcal{C}$ to sets.

16. The inclusion functors

$$I_1\colon \mathbf{abelian\ groups} \to \mathbf{groups}$$
$$I_2\colon \mathbf{Hausdorff\ topology} \to \mathbf{topology}$$
$$I_3\colon \mathbf{comrings} \to \mathbf{rings}$$

preserve products and pullbacks.

17. (Basic.) If $\Phi, \Psi: \mathcal{A} \to \mathcal{B}$ are two functors, then their product in the CCategory **natural**$(\mathcal{A}, \mathcal{B})$ has the property that, for each $\mathcal{A}$-object A, it is the case that $(\Phi \times \Psi)(A) = (\Phi(A)) \times (\Psi(A))$. Similarly, the coproduct is described in terms of the coproduct in $\mathcal{B}$. This generalizes to arbitrary limits and colimits in **natural**$(\mathcal{A}, \mathcal{B})$.

18. As **13** showed, a commutative diagram **D** in $\mathcal{A}$ is a functor from a preorder Γ to $\mathcal{A}$. Preorders form a full subCategory of the Category of categories and functors; in the Category of preorders, products exist; in particular, if Γ and Δ are preorders with relations $\preceq_\Gamma$ and $\preceq_\Delta$, then $\Gamma \times \Delta$ consists of the Cartesian product of the underlying sets of objects, with the relation $\preceq$ defined by

$$(a, b) \preceq (c, d) \text{ iff both } a \preceq_\Gamma c \text{ and } b \preceq_\Delta d.$$

Now, let **D** be a commutative diagram in $\mathcal{A}$ based on the preorder Γ, and let **E** be a commutative diagram in **natural**$(\mathcal{A}, \mathcal{B})$ based on the preorder Δ. Then put these two together to obtain a commutative diagram **ED** in $\mathcal{B}$ modeled on the preorder $\Delta \times \Gamma$. The basic meaning of this is that, out of commutative diagrams and natural maps, one can "naturally" obtain **even more commutative diagrams**.

19. If R is a right adjoint and α is monic, then $R\alpha$ is monic. Dually, a left adjoint preserves epic.

20. Suppose $L: \mathcal{B} \to \mathcal{A}$ and $R: \mathcal{A} \to \mathcal{B}$ are adjoint, so that

$$\mathcal{A}[L(B), A] \approx \mathcal{B}[B, R(A)].$$

Then to the identity map $L(B) \to L(B)$ there corresponds a map

$$\varepsilon_B: B \to RL(B).$$

This ε is a natural map from (the identity functor on) $\mathcal{B}$ to RL. Given $\varphi: L(B) \to A$ in $\mathcal{A}$, apply R to get $R\varphi: RL(B) \to R(A)$, and compose with ε_B to get

$$(R\varphi)\varepsilon_B: B \to R(A).$$

This is the $\mathcal{B}$-map corresponding to φ under the natural equivalence

$$\mathcal{A}[L(B), A] \approx \mathcal{B}[B, R(A)].$$

In other words, this natural equivalence is determined by the natural transformation ε. Call ε the "**unit**" of the adjoint pair (L, R).

Dually, there is a counit $\eta: LR \to \mathcal{A}$, which determines the natural equivalence.

21. Let $L: \mathcal{A} \to \mathcal{B}$ and $R: \mathcal{B} \to \mathcal{A}$ be such that (L, R) is a pair of adjoint functors. Suppose that $RL: \mathcal{A} \to \mathcal{A}$ is naturally equivalent to the identity functor on $\mathcal{A}$. Let **D** be a commutative $\mathcal{A}$-diagram. Suppose that $\boldsymbol{\beta}: B \to L(\mathbf{D})$ is the limit in $\mathcal{B}$ of $L(\mathbf{D})$. Then $R\boldsymbol{\beta}: R(B) \to RL(\mathbf{D})$, composed with a natural equivalence from RL to the identity on $\mathcal{A}$, describes $R(B)$ as the $\mathcal{A}$-limit of **D**. Dualize.

The category of sets.

22. In a preorder, every map is both monic and epic. An initial object is a minimum element, and a terminal object is a maximum element.

23. In the Category of sets, the empty set $\emptyset$ is the initial object, and singleton sets are the terminal objects. A map is monic if and only if it is one-to-one (= **injective**). A map is epic if and only if it is onto (= **surjective**). A map which is both monic and epic is an equivalence (= **bijective**). If $\alpha\colon A \to B$ is monic and $A \neq \emptyset$, then α is a coretraction. The **axiom of choice** says that every epic map in this Category is a retraction.

23. In the Category of sets, product = Cartesian product of sets, decorated with the coordinate projections. The pullback of (α, β), where $\alpha\colon A \to B$ and $\beta\colon C \to B$, is

$$P = \{\, (a, c) \in A \times C \;:\; \alpha(a) = \beta(c) \,\}.$$

The inverse limit of an inverse system can be described as a certain subset of the Cartesian product of the sets at the vertices of the diagram.

24. Let $R \subset X \times X$, where R and X are sets. Let $\alpha, \beta\colon R \to X$ be the restrictions of the two coordinate projections. The coequalizer of (α, β) in the Category of sets is exactly what is meant by the "set of **equivalence classes** of the equivalence relation on X generated by R". In general, colimits in the Category of sets can be described as such "quotient sets" of coproducts (= disjoint union) of sets.

25. The Product of *all* sets is a Set; namely, it is the **empty Set**. The Coproduct of all sets, however, is a very big Set.

Groups and monoids.

26. In the Category of monoids, there is an object which is both initial and terminal.

27. In the Category of groups, the existence, for every map $\varphi\colon G \to H$ of a subgroup of G, the kernel $K = \{\, g \in G \;:\; \varphi(g) = 1 \,\}$, can be used to show that every monic group-map is injective.

28. In the Category of groups, every epic group-map is surjective. This is difficult, and requires some ingenuity to prove. (Try this yourself, and then look at **32**.)

29. It is fairly easy, however, to show that in the Category of abelian groups, every epic map is surjective. This is also easy to show in the Category of Λ-modules.

30. The underlying set of a group G is a G-set in an obvious way. Given any G-set X and $x \in X$, there is a map of G-sets $\epsilon_x\colon \text{set } G \to X$, sending g to gx. For G-sets

X and Y, with $x \in X$ and $y \in Y$, describe the pushout of the diagram

$$X \overset{\epsilon_x}{\leftarrow} \text{set } G \overset{\epsilon_y}{\rightarrow} Y.$$

This gives an idea of how pushouts in the Category of G-sets can be described, in general.

31. If X is a G-set and $x \in X$, there are the **orbit** Gx and the **stabilizer** $S(x) = \{g \in G : gx = x\}$. The set X is the disjoint union of the orbits, and each orbit is itself a G-set; thus, every G-set is the coproduct, in the G-set Category, of certain elementary G-sets. — To describe these elementary G-sets, consider a subgroup S of G. The set G is itself an S-set, by using right multiplication, as follows: For $s \in S$ and $z \in G$, define $s * z = zs^{-1}$; the set of orbits of the set G as an S-set is called the set of **left cosets** of S in G, and is denoted by G/S; the orbit containing z is denoted, suggestively, by zS. Now, G acts on G/S by left multiplication in the obvious way, and makes G/S into a G-set. — Given a G-set X and $x \in X$, there are the stabilizer $S(x)$ which is a subgroup of G and the orbit Gx, which is itself a G-set. Each left coset $zS(x)$ determines the element zx of Gx, which is independent of the choice of z in that left coset. This describes an equivalence in the G-set Category, from $G/S(x)$ to Gx. To sum up, here is the structure theorem for G-sets: Every G-set is the coproduct of elementary G-sets of the form G/S.

32. We are now in position to solve the problem posed in **28**. The point is that if S is a proper subgroup of G (in the problem, S corresponds to the image of a group-map which one wants to show is not epic), then construct a set X, consisting of G/S plus an extra element x_0. Make X into a G-set in two different ways: The first way, X_1, contains the orbit G/S and a singleton orbit $\{x_0\}$. The second way, X_2, interchanges the roles of x_0 and the coset $1S$. Each of these actions yields a group-map from G to the group of set-self-equivalences of X; the equalizer of these two group-maps (φ_1, φ_2) is exactly the subgroup S. Now finish the problem and show that if α is a group-map with target G and image S, then $\varphi_1\alpha = \varphi_2\alpha$, and thus (under the assumption that S is a proper subgroup of G) conclude that α is not epic in the category of groups.

33. The product, pullback, and inverse limit notions in the Category of groups are very similar to those in the Category of sets. In particular, the product is the direct product of groups. On the other hand, colimits in the Category of groups are a bit more subtle and interesting to describe; for instance, the coproduct in the Category of groups is the construction called "free product".

34. In the Category of abelian groups, the coproduct is the "**direct sum**". And then, in this Category, the other colimits are easily described in terms of quotient groups of direct sums. A similar thing holds true in the Category of Λ-modules. In particular, suppose that

$$A \overset{\alpha}{\leftarrow} B \overset{\beta}{\rightarrow} C$$

is a diagram in the Category of Λ-modules. Its pushout can be described as the quotient module P of $A \times C$ by the submodule consisting of

$$\{\,(\alpha(b), -\beta(b))\ :\ b \in B\,\}.$$

The map $A \to P$ takes a to $(a, 0)$, and the map $C \to P$ takes c to $(0, c)$. There is something a little bit odd about this situation and this notation, because, in the Category of Λ-modules, there is an equivalence from $A \sqcup C$ to $A \sqcap C = A \times C$. Such a map exists in this Category by virtue of the existence of an object $\{0\}$ which is both terminal and initial; the fact that it is an equivalence is an odd fact about this kind of Category.

35. (This is a simple but deep factlet, perhaps out of place here, which is an exercise in "the fine print".) Let $X \subset G$, where G is a group, and X is a subset. Then consider the class of all subgroups of G which contain X; this forms an inverse system, whose limit is identifiable as the intersection of all these subgroups; it is the smallest subgroup containing X. It is called the subgroup **generated by** X; if this is the whole group G, then X is called a set of generators of G. — Now, suppose that $\varphi: G \twoheadrightarrow H$ is a surjective group-map; if X is a set of generators of H, it is not always true that $\varphi^{-1}(X)$ is a set of generators of G. However, there is a tiny extra assumption which does make this true.

36. (Somewhat subtle and difficult.) In the Category of perfect groups, the product has a certain subtlety. Find an example of a sequence of perfect groups, whose product in the Category of groups is not perfect!

Now, in any given group G, the union of the perfect subgroups of G generates a subgroup which can be easily seen to be perfect. This maximum perfect subgroup of G will be called the **perfection** of G. The product, in the Category of perfect groups, of a family of perfect groups, is the perfection of the direct product (= product in the Category of groups) of that family of groups.

37. The inclusion functor J: **perfect groups** $\to$ **groups** does not preserve products. It does not even preserve pullbacks, because it is possible to have two perfect subgroups of a perfect group, whose intersection is not perfect.

38. The "commutator subgroup" is a functor from groups to groups. Find what the commutator subgroup of the direct product of two groups is; and ask if this result extends to the product of infinitely many groups. Show that this functor preserves epic, but not monic.

39. Perfection yields a functor from groups to perfect groups. This functor preserves product.

40. If G is a group, let G' denote its commutator subgroup. Then the quotient group G/G' is an abelian group, called the **abelianization** of G. Abelianization is a functor from groups to abelian groups; it may not be perfectly obvious, but it is true that abelianization preserves coproduct, pushout, and all colimits. Abelianization also preserves finite products; it does not preserve pullbacks or infinite products.

41. The subject matter of **36** to **40** can be considerably generalized. The notion of commutator subgroup of a group G can be imagined to be related to the expression $xyx^{-1}y^{-1}$; substitute all possible elements of G in for x and y, and the elements of G obtained thus generate the commutator subgroup of G; a perfect group is one which equals its commutator subgroup; the abelianization of G is the quotient group of G by its commutator subgroup. Now, instead of $xyx^{-1}y^{-1}$, one can take some other expression, and similar things can be discussed. For instance, using x^{667}, one gets the subgroup generated by all 667-th powers, the kind of group which equals this 667-th power subgroup, and the quotient group, the largest quotient group which is of exponent 667. Or $xyzy^{-1}z^{-1}x^{-1}zyz^{-1}y^{-1}$, which would define groups which are "nilpotent of class 2". — Find another group theory book that contains the terms "verbal subgroup" and "variety of groups", and relate them to this.

42. Consider the Category of groups. Every limit has a fairly elementary construction as a subgroup of a direct product of groups; thus, limits exist in this Category. Now, let us try to construct the coproduct of two groups A and B; if $\alpha: A \to X$ and $\beta: B \to X$ are two group-maps, then the subgroup of X generated by $\alpha(A) \cup \beta(B)$ is not arbitrarily huge; in fact, it can be shown to have cardinality bounded by the maximum of $\aleph_0$ and $\aleph =$ cardinality$(A \cup B)$. Thus, there exists a (small) set of pairs of group-maps (α, β), where $\alpha: A \to X_{\alpha\beta}$ and $\beta: B \to X_{\alpha\beta}$, such that the nature of the coproduct $A \sqcup B$ can be described by testing it against all such pairs (α, β). The property of products in the Category of groups then yields a pair of group-maps

$$\prod_{(\alpha,\beta)} \alpha: A \to \prod_{(\alpha,\beta)} X_{\alpha\beta} \qquad \prod_{(\alpha,\beta)} \beta: B \to \prod_{(\alpha,\beta)} X_{\alpha\beta}$$

Then, the coproduct $A \sqcup B$ can be identified with the subgroup of $\prod X_{\alpha\beta}$ generated by the union of the images of $\prod \alpha$ and $\prod \beta$.—This, of course, does not give much of a feeling for the coproduct; this will come later. By the way, the classical terminology is to call the coproduct of groups their **free product**, and to write it $A * B$.

43. Consider the Category of groups. There exist the identity functor id: **groups** $\to$ **groups**, the commutator subgroup functor

$$\text{csgp}: \textbf{groups} \to \textbf{groups},$$

and the abelianization functor

$$\text{ab}: \textbf{groups} \to \textbf{groups}.$$

There are natural maps

$$\text{csgp} \xrightarrow{incl} \text{id} \xrightarrow{quot} \text{ab}$$

where, for each group G, there are group-maps *incl*, the inclusion of the commutator subgroup G' into G, and *quot*, the quotient group-map from G to G/G'.

44. The duality functor *: **groups** $\to$ **groups** is naturally equivalent to the identity functor via $E_G: G \to G^*$ given by $E_G(g) = (g^{-1})^*$.

45. The forgetful functor Φ from groups to sets preserves limits. It has a left adjoint F, so that

$$\mathbf{groups}[F(X), G] \approx \mathbf{sets}[X, \Phi(G)].$$

Call $F(X)$ the **free group** on the set X. The explicit construction of F can be made in much the same spirit that free products were constructed in **42**. The unit phenomenon is exactly the "universal property" generally ascribed to free groups. That is, there is a function ε_X from X to the underlying set of $F(X)$, such that for any group G, a function $f: X \to$ underlying set of G determines a unique group-map $\varphi: F(X) \to G$ such that f is the composition of ε_X with φ considered as a set-map.

46. The inclusion functor I: **abelian groups** $\to$ **groups** and the abelianization ab: **groups** $\to$ **abelian groups** form an adjoint pair (ab, I). This explains, for instance, why abelianization preserves pushout and coproduct, and helps give an understanding that it is unlikely that abelianization preserves product and pullback. Find interesting examples.

Rings and modules.

47. In the Category of rings, the ring of integers $\mathbf{Z}$ is an initial object, and the zero-ring $\{0\}$ is a terminal object. (Note that in the definition of ring, it was not assumed that $1 \neq 0$.) The inclusion ring-map $\mathbf{Z} \to \mathbf{Q}$ is epic, where $\mathbf{Q}$ denotes the ring of rational numbers. A ring-map whose kernel is non-zero is not monic; to prove this requires a clever construction (left to the reader).

48. In the Category of rings, the product of a family of rings is simply the Cartesian product of the underlying sets, in which both the binary operations are defined coordinate-wise. The coproduct, on the other hand is difficult to describe in explicit terms.

49. For any ring Λ, define

$$U(\Lambda) = \{\, \lambda \in \Lambda : \exists\, \mu \in \Lambda \ni \mu\lambda = \lambda\mu = 1 \,\}.$$

This is a group, the group of **units of** Λ. This gives a functor

$$U : \mathbf{rings} \to \mathbf{groups}$$

which preserves product and pullback.

50. Let G be a group. Then $\mathbf{Z}G$ is the **group ring** of G with coefficients in the ring of integers $\mathbf{Z}$. It is the set of finite linear combinations of elements of G with coefficients in $\mathbf{Z}$, with the operation of addition defined in the obvious way and the operation of multiplication defined using the binary operation in G in such a way that the distributive laws hold. This defines a functor from groups to rings.

51. Let M be a $\mathbf{Z}G$-module. Forgetting the additive structure yields a forgetful functor from $\mathbf{Z}G$-modules to G-sets.

52. Let X be a G-set. The set of finite linear combinations with integer coefficients, of elements of X has an obvious structure of $\mathbf{Z}G$-module. This gives a functor from G-sets to $\mathbf{Z}G$-modules.

53. The concept of $n \times n$-matrix yields a functor

$$\mathbf{M}_n\colon \mathbf{rings} \to \mathbf{monoids}.$$

This functor preserves product.

54. Consider G-sets and $\mathbf{Z}G$-modules. There is a forgetful functor $\Phi\colon \mathbf{Z}G\textbf{-modules} \to G\textbf{-sets}$, and (described in **52**) the functor $\mathbf{Z}\colon G\textbf{-sets} \to \mathbf{Z}G\textbf{-modules}$. If X is a G-set and M a $\mathbf{Z}G$-module, then

$$G\textbf{-sets}[X, \Phi(M)] \approx \mathbf{Z}G\textbf{-modules}[\mathbf{Z}X, M].$$

Thus, the functor $\mathbf{Z}$ preserves colimits. This is a very useful fact, because, for example, as was pointed out in **30**, pushouts in the Category of G-sets are rather hard to contemplate, whereas pushouts in the Category of $\mathbf{Z}G$-modules are easily described in terms of direct sum and quotient modules.

55. The group of units functor $U\colon \mathbf{rings} \to \mathbf{groups}$ has as its left adjoint the group-ring functor $\mathbf{groups} \to \mathbf{rings}$ taking G to $\mathbf{Z}G$. The group-ring functor therefore preserves coproduct, giving some concrete examples of coproducts of rings.

Topology.

56. In the topology Category, there is a two-point discrete space (four open sets), and a two-point indiscrete space (two open sets). Using these to test with, prove that every monic topology-map is injective (*i.e.*, monic on the underlying set) and that every epic topology-map is surjective. Not every surjective topology-map is a retraction, nor is every injective topology-map a coretraction. An equivalence in the topology Category is called a **homeomorphism**. There are maps which are both monic and epic which are not homeomorphisms.

57. A **closed subspace** is one whose complement is open. The **closure** in X of a subset $S \subset X$, is the intersection of all closed sets which contain it.

In the Hausdorff topology Category, every monic map is injective. A map, the closure of whose image is the target space, is epic. This characterizes epic maps in this Category, and this can be proved by constructing, given a closed subset C of a Hausdorff space X, the union of two copies of X with the two copies of C identified; the resulting space is Hausdorff, and it can be used for testing whether a map to X whose image has closure C is epic.

In Hausdorff topology, a coretraction $\alpha\colon A \to B$ has the property that the image of α is a closed subspace of B, and that, if α' denotes the set-map α with its target restricted to this image, then α' is a homeomorphism. Neither of the claims in the preceding sentence is true in the (non-Hausdorff) topology Category.

58. In the compact Hausdorff topology Category, monic equals injective and epic equals surjective. Furthermore, epic plus monic implies equivalence.

59. In topology, the product is the product of the underlying family of sets, endowed with the minimum possible class of sets satisfying the axioms for open sets, such that the coordinate projections are continuous. This is the usual, or "Tychonoff" topology of the product. The pullback is the same as in the Category of sets, with the topology of a subspace of the product; in fact, the principal **rationale for the subspace topology** is this relationship to pullbacks.

In topology, the coproduct is the "disjoint open union". The pushout of a diagram is a certain topology on the set-theoretic pushout, which is the rationale for the notion of the quotient topology.

60. In Hausdorff topology, products, pullbacks, and coproducts are exactly as in the topology Category. However, the pushout of a Hausdorff topology diagram, in the topology Category, may not be Hausdorff. It is worth some thought to imagine how pushouts can be described in Hausdorff topology. For instance, an identification space of a Hausdorff space need not be Hausdorff; but there is a largest continuous image of this which is Hausdorff.

61. The product, in the topology Category, of two k-spaces may not be a k-space. Work on finding an example (you can find some complicated examples by considering CW-complexes; every CW-complex is a k-space, but the product of two CW-complexes may not have, in the topology category, the structure of a CW-complex). This can be fixed up fairly easily, as follows: Given a Hausdorff space X, one can define a topology on the same underlying set which may have more open sets, by declaring a set to be open in the new topology $k(X)$ if and only if its intersection with every compact subspace of X is open in that subspace. Then the k-space product is obtained by applying the process "k" to the topological product.

62. It is worth thinking about what the coproduct of an infinite number of compact Hausdorff spaces is in the Category of compact Hausdorff spaces. For instance, take a countable number of 1-point spaces $\{\{x_i\} : i = 1, 2, \ldots, n \ldots\}$; then the coproduct E in the Category of compact Hausdorff spaces has to be a compact Hausdorff space that contains a countable discrete set as a dense subspace, and to have the property that every continuous map from this countable discrete set into any compact Hausdorff space Y (i.e., any sequence in the space Y) extends to a continuous fuction E to Y. In some sense, therefore, each point of E, via its image in Y, will give some kind of "limit" for *any* sequence of points of Y. If you take a point $e \in E$, which does not belong to the original discrete subspace, and if the image in Y, $\{y_1, \ldots, y_n, \ldots\}$, does in fact converge to a point of Y, then the limit depending on e will be the actual limit of that sequence. We can imagine describing E by mapping the discrete countable set X into All compact Hausdorff spaces, taking the Product, and looking at the closure of the image of X in this Product; and reducing the case to a space instead of a Space. This E is the **Stone-Čech compactification** of X; and the limit of an arbitrary sequence in a compact Hausdorff space depending on e, is an **ultralimit**.

63. In sets, topology, groups, and rings, inverse and direct limits behave very much as they do in the Category of sets. In the topology case, one needs to say a little bit about how the open sets of the direct limit are defined. In compact Hausdorff topology, direct limits behave strangely. Can you find an example of a direct system of Hausdorff spaces whose direct limit in the topology Category is not Hausdorff? In the Categories of k-spaces and perfect groups, inverse limits have to be fixed up.

64. The inclusion functor from k-spaces to Hausdorff spaces does not preserve products. (This is analogous, in a funny way, to the fact that the inclusion functor from perfect groups to groups does not preserve products.)

65. The k-operation yields a functor from Hausdorff spaces to k-spaces, which preserves products. (Thus, k is analogous to "perfection".)

66. Generalize the discussion in **62** about the Stone-Čech compactification of a discrete space, to the compactification of any Hausdorff space. And thus, describe how to construct the coproduct of an arbitrary set of compact Hausdorff spaces, in the Category of such spaces. Note how one uses the facts that the product of Hausdorff spaces is Hausdorff, the product of compact spaces is compact, and that there is only a set (not a Set) of closures of images of a given space in various compact Hausdorff spaces.

67. Consider the Category of Hausdorff spaces; every limit in this Category can be described as a subspace of a product of Hausdorff spaces; this shows that all limits exist. Now, continuing **60**, try to construct the pushout of a diagram of Hausdorff spaces, in the Hausdorff Category; map the diagram into a huge product of Hausdorff spaces, by taking the product of a sufficiently big set of maps of the diagram into Hausdorff spaces; the pushout is then the subspace of the product consisting of the union of the images of the objects in the diagram.

68. The forgetful functor from **topology** to **sets** has a left adjoint ("**discrete space**") and a right adjoint ("**indiscrete space**").

69. Let X be a topological space. Consider all conceivable Hausdorff images of X. That is, consider all equivalence relations on the underlying set of X, all Hausdorff topologies on these quotient sets, and only those for which the quotient function is continuous. This makes a set of topology maps $\alpha: X \to S_\alpha$, where S_α is Hausdorff. Thus there is $\prod \alpha: X \to \prod S_\alpha$, and the image of $\prod \alpha$ is the "largest" Hausdorff image of X. Call this $H(X)$, the "**Hausdorffification**" of X.

Now, this extends to a functor H from topology to Hausdorff topology. It is related to the inclusion functor J from Hausdorff topology by an adjointness relation **topology**$[X, J(A)] \approx$ **Hausdorff topology**$[H(X), A]$. Note that the fact that J preserves limits was used. The argument outlined here is essentially how the proof of the adjoint functor theorem goes.

70. Here is a (non-commutative) diagram of functors:

$$\textbf{k-spaces} \underset{k}{\overset{I}{\rightleftarrows}} \textbf{Hausdorff topology} \underset{H}{\overset{J}{\rightleftarrows}} \textbf{topology}$$

Here, I and J are inclusion functors, H is the Hausdorffification described in the preceding exercise, and k is the functor described in **61**. Note that kI and HJ are (naturally equivalent to) the identity functors on k-spaces and Hausdorff, respectively. As above, (H, J) is an adjoint pair.

It is also true that (I, k) is an adjoint pair; this says that if A is a k-space and B a Hausdorff space, then if $f: A \to k(B)$ is continuous, so is the same function as a function from $A = I(A)$ to B; that happens because (1) a function with source a k-space A is continuous if and only if it is continuous on each compact subspace of A; (2) the continuous image of a compact space is compact; and (3) the compact subspaces of B and of $k(B)$ are the same. A consequence is that k preserves limits; hence, since kI is the identity and Hausdorff topology is complete, it follows that the Category of k-spaces is complete.

Now, let $\mathbf{D}$ be a diagram of k-spaces, with colimit A. Since I is a left adjoint, then $I(A)$ is the colimit of the Hausdorff diagram $\mathbf{E} = I(\mathbf{D})$. Since HJ is the identity on Hausdorff topology, and H is a left adjoint, it follows from **21**, that the Hausdorff colimit of $\mathbf{E}$ is H of the topological colimit of $J\mathbf{E}$. This implies that $I(A) = H(\operatorname{colim} JI(\mathbf{D}))$.

In particular, identification spaces are colimits. This result shows that if one starts with a k-space A and wishes to identify a subspace A' via a map $\varphi: A' \to A''$, and if one then creates the identification space in the topology Category, and if the result turns out to be Hausdorff, then the result is in fact a k-space. (And, in any case, the Hausdorffification of a topological identification space of a k-space is a k-space.) This "explains" why CW-complexes, defined to be topological identification spaces, are k-spaces.

71. The inclusion functor

$$I\colon \textbf{compact Hausdorff} \to \textbf{topology}$$

preserves limits. Therefore, one expects the existence of a left adjoint

$$\check{C}\colon \textbf{topology} \to \textbf{compact Hausdorff}$$

definable by checking through the proof of the adjoint functor theorem; map the topological space X into the product of all conceivable compact Hausdorff spaces; look at the closure of the image; show that this Space is topology-equivalent to an ordinary space; and call it $\check{C}(X)$. This functor $\check{C}$ is, more or less, what is called the **Stone-Čech compactification**. The unit for the adjointness relation is a topology-map $\varepsilon_X\colon X \to \check{C}(X)$, which satisfies a universal property of some interest. The standard definition of Stone-Čech compactification limits the target of the inclusion functor to the full subcategory of those topological spaces for which ε_X is an embedding as a subspace; that is, to completely regular Hausdorff spaces.

References

[1.] Mac Lane, S., *Categories for the Working Mathematician*, Graduate Texts in Mathematics 5, Springer (1971).

[2.] Freyd, P., *Abelian Categories*, Harper and Row (1964) [Adjoint Functor Theorem].

[3.] Freyd, P., Scedrov, Andre, *Categories, Allegories*, North-Holland mathematical library 39, Elsevier (1990).