Grothendieck Rings, Euler Characteristics, and Schanuel Dimensions of Models

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29 September 2000

$$\chi = F - E + V$$

Interpretation of χ

- Euler-Poincaré characteristic: $\chi(X) = \sum(-1)^i \dim H^i(X)$
- (hyper-)graph theoretic/combinatorial version
- additive invariant of definable sets

O-minimal structures

Definition 1 *A linearly ordered structure $\mathcal{M} = (M, <, \cdots)$ in a language extending the language of ordered sets is* o-minimal *if every (parametrically) definable subset of $\mathcal{M}$ is a finite Boolean combination of points and intervals of the form (a, b) with $a, b \in M \cup \{-\infty, \infty\}$.*

Examples:

- $(\mathbb{Q}, <)$
- $(D, < .+, \{\lambda\cdot\}_{\lambda\in D})$ where D is an ordered division ring.
- $(\mathbb{R}, <, +, \cdot, 0, 1)$
- $(\mathbb{R}, <, +, \cdot, \exp, 0, 1)$

Abstract Euler characteristics

Definition 2 *An* Euler characteristic *on a first-order structure $\mathcal{M}$ is a function χ from the set of (parametrically) definable subsets of powers of $\mathcal{M}$ to some ring satisfying*

- $\chi(X) = \chi(Y)$ *if there is a definable bijection between X and Y,*
- $\chi(X \dot{\cup} Y) = \chi(X) + \chi(Y)$,
- $\chi(X \times Y) = \chi(X) \cdot \chi(Y)$, *and*
- $\chi(\{*\}) = 1$ *for any* $* \in \mathcal{M}$.

Euler characteristics on o-minimal structures

Theorem 3 (van den Dries) *If* $\mathcal{M} = (M, <, +, \cdot, 0, 1, \ldots)$ *is an o-minimal expansion of an ordered field, then there is a unique Euler characteristic* χ *on* $\mathcal{M}$ *with values in* $\mathbb{Z}$*. Moreover, if the underlying field is* $\mathbb{R}$*, then* χ *agrees with the topological Euler characteristics on definable manifolds.*

The o-minimal Euler characteristic is a finer invariant than the topological Euler characteristic. For example, $\chi_0((0, 1)) = -1 \neq 0 = \chi_0([0, 1))$ but $\chi_{\text{top}}((0, 1)) = -1 = \chi_{\text{top}}([0, 1))$.

The rig of definable sets

Definition 4 *Given an $\mathcal{L}$-structure $\mathcal{M}$ and a natural number n,* $\mathrm{Def}^n(\mathcal{M})$ *is the set of all $\mathcal{L}_M$-definable subsets of of $\mathcal{M}^n$. The set* $\mathrm{Def}(\mathcal{M})$ *is* $\bigcup\limits_{n=0}^{\infty} \mathrm{Def}^n(\mathcal{M})$.

$\mathrm{Def}(\mathcal{M})$ forms a category with the morphisms between two definable sets being the set of definable functions between them.

Definition 5 $\widetilde{\mathrm{Def}}(\mathcal{M})$ *is the set of isomorphism classes of definable subsets of powers of $\mathcal{M}$. We write* $[\ ] : \mathrm{Def}(\mathcal{M}) \to \widetilde{\mathrm{Def}}(\mathcal{M})$ *for the map which associates to a definable set its isomorphism type.* $\widetilde{\mathrm{Def}}(\mathcal{M})$ *carries a natural $\mathcal{L}_{\mathrm{ring}}$-structure with* $[X]+[Y] := [X \dot\cup Y]$, $[X]\cdot[Y] := [X \times Y]$, $0 := [\varnothing]$, *and* $1 := [\{*\}]$.

Rigs

$\widetilde{\mathrm{Def}}(\mathcal{M})$ is a *rig* or *semiring*, but is never a ring as, for instance, $\widetilde{\mathrm{Def}}(\mathcal{M}) \models 0 \neq 1 \;\&\; (\forall\, x, y)\; x + y = 0 \;\rightarrow\; x = y = 0$.

Definition 6 *A* rig *(or a* semiring*) is an $\mathcal{L}_{\mathrm{ring}}$-structure for which*

- $+$ *is a commutative, associative operation with null element* 0*,*
- $\cdot$ *is an associative operation with null element* 1*,*
- *left- and right-multiplication by* 0 *are the zero function, and*
- $\cdot$ *is left- and right-distributive over addition.*

The rig is commutative *if multiplication is also a commutative operation.*

Axioms for $\mathrm{Def}(\mathcal{M})$

The rig $\mathrm{Def}(\mathcal{M})$ satisfies

- $0 \neq 1$
- $(\forall x, y)\ x \cdot y = y \cdot x$ (commutativity)
- $(\forall x, y)\ x + y = 0 \to x = 0 = y$
- $(\forall x, y)\ x \cdot y = 1 \to x = 1 = y$
- $(\forall x_1, x_2, y_1, y_2)(\exists z_{1,1}, z_{1,2}, z_{2,1}, z_{2,2})$
  $[x_1 + x_2 \to \bigwedge_{i=1}^{2}(x_i = z_{i,1} + z_{i,2} \ \&\ y_i = z_{1,i} + z_{2,i})]$

Question 1 What is $\mathrm{Th}_{\mathcal{L}_{\mathrm{ring}}}(\{\widetilde{\mathrm{Def}}(\mathcal{M}) : \mathcal{M}$ a first-order structure $\})$?

The Grothendieck ring

Given a rig $(R, +, \cdot, 0, 1)$, there is a universal morphism from R to a ring.

On $R \times R$ define $(a, b) \sim (c, d) \Leftrightarrow (\exists z \in R)\ a + d + z = c + b + z$.

The quotient $\mathcal{R}(R) := (R \times R)/\sim$ is a ring and the map $R \to \mathcal{R}(R)$ given by $x \mapsto [(x, 0)]_{\sim}$ (="$x - 0$") is a rig morphism.

In general, the morphism $R \to \mathcal{R}(R)$ need not be injective.

Definition 7 *A (weak) Euler characteristic on $\mathcal{M}$ is a $\mathcal{L}_{\mathrm{ring}}$-morphism $\chi : \widetilde{\mathrm{Def}}(\mathcal{M}) \to R$ where R is a ring.*

Definition 8 *The Grothendieck ring of a first-order structure $\mathcal{M}$ is $K_0(\mathcal{M}) := \mathcal{R}(\widetilde{\mathrm{Def}}(\mathcal{M}))$. The ringification map $\chi_0 : \widetilde{\mathrm{Def}}(\mathcal{M}) \to K_0(\mathcal{M})$ is the universal (weak) Euler characteristic on $\mathcal{M}$.*

Aside on distributive categories

Definition 9 *A* distributive category *is a category* $\mathcal{C}$ *with an initial object* $\perp$*, a final object* $\top$*, finite limits and finite colimits, and for which the natural morphism* $(A \times C) \coprod (A \times B) \to A \times (B \coprod C)$ *is an isomorphism for any* $A, B, C \in \mathrm{Ob}(\mathcal{C})$.

For any small distributive category $\mathcal{C}$, the set of isomorphism classes of objects forms a rig $R(\mathcal{C})$. The rig of model $\mathcal{M}$ is the special case of $\mathcal{C} = \mathrm{Def}(\mathcal{M})$.

Question 2 Are the theories $\mathrm{Th}_{\mathcal{L}_{\mathrm{ring}}}(\{R(\mathcal{C}) : \mathcal{C}$ a small distributive category with $[\perp] \neq [\top]\})$ and $\mathrm{Th}_{\mathcal{L}_{\mathrm{ring}}}(\{\widetilde{\mathrm{Def}}(\mathcal{M}) : \mathcal{M}$ a first-order structure $\})$ the same?

Examples of $K_0(\mathcal{M})$

- If $\mathcal{M}$ is a finite structure, then $\widetilde{\mathrm{Def}}(\mathcal{M}) \cong \mathbb{N}$ with the map being $[X] \mapsto \|X\|$. The Grothendieck ring is $\mathbb{Z}$ and every Euler characteristic is given by counting modulo some integer.

- If $\mathcal{M} = (\omega, S)$, then $K_0(\mathcal{M}) = 0$ as the decomposition $\varnothing \dot\cup \omega = \{0\} \dot\cup S(\omega)$ and the definable isomorphism $S : \omega \to S(\omega)$ yield $0 + [\omega] = 1 + [\omega]$ in $\widetilde{\mathrm{Def}}(\mathcal{M})$ and hence $0 = 1$ in $K_0(\mathcal{M})$. However, $\widetilde{\mathrm{Def}}(\mathcal{M})$ is more complicated.

More examples of $K_0(\mathcal{M})$

- If $\mathcal{M}$ is an o-minimal expansion of a field, then $K_0(\mathcal{M}) = \mathbb{Z}$.
- If $\mathcal{M} = (\mathbb{C}, +, \cdot, 0, 1)$, then $K_0(\mathcal{M})$ is very complicated. At the very least, the universal Euler characteristic on $\mathbb{R}$ induces an Euler characteristic on $\mathbb{C}$ as $\mathbb{C}$ is interpretable in $\mathbb{R}$.
- $K_0(\mathbb{Q}, <)$ embeds in $\mathbb{Q}[\{X_a\}_{\{a \in \mathbb{Q} \cup \{\infty\}\}}]$ as the subring of numerical polynomials. (Matthew Frank)

Strong Euler characteristics

Definition 10 *A* strong Euler characteristic $\chi : \widetilde{\mathrm{Def}}(\mathcal{M}) \to R$ *on a structure* $\mathcal{M}$ *is an Euler characteristic satisfying the fibration condition:*

If $\pi : E \to B$ *is definable function between definable sets,* $f \in R$*, and for all* $b \in B$ *one has* $\chi([\pi^{-1}\{b\}]) = f$*, then* $\chi([E]) = f \cdot \chi([B])$.

The fibration condition differs from the other axioms for an Euler characteristic in two important respects:

- It is not rig-theortetic.
- In any reasonable language it is syntactically more complicated than the other axioms.

Proposition 11 *On any structure* $\mathcal{M}$ *there is a universal strong Euler characteristic* $\chi^s : \widetilde{\mathrm{Def}}(\mathcal{M}) \to K^s(\mathcal{M})$.

Examples of strong Euler characteristics

- The universal weak Euler characteristics on finite structures and o-minimal expansions of fields are strong.
- More generally, if $\mathcal{M}$ is a structure with the property that every definable function is a locally trivial fibration, then every Euler characteristic on $\mathcal{M}$ is strong.
- The universal weak Euler characteristic on $\mathbb{C}$ is *not* strong.
- Every strong Euler characteristic on an algebraically closed field of positive characteristic is trivial.

Dependence on the theory

Theorem 12 *If* $\mathcal{M} \equiv \mathcal{N}$, *then* $\widetilde{\mathrm{Def}}(\mathcal{M}) \equiv_{\exists_1} \widetilde{\mathrm{Def}}(\mathcal{N})$.

Proof:

- If $\mathcal{U}$ is an ultrafilter, then $\mathrm{Def}(\mathcal{M}) \subseteq \mathrm{Def}(\mathcal{M}^{\mathcal{U}}) \subseteq \mathrm{Def}(\mathcal{M})^{\mathcal{U}}$ while the inclusion $\mathrm{Def}(\mathcal{M}) \subseteq \mathrm{Def}(\mathcal{M})^{\mathcal{U}}$ is elementary in the full language of $\mathrm{Def}(\mathcal{M})$.
- Thus, $\mathrm{Def}(\mathcal{M}) \preceq_{\exists_1} \mathrm{Def}(\mathcal{M}^{\mathcal{U}})$.
- As $\widetilde{\mathrm{Def}}(\mathcal{M})$ is existentially interpretable in $\mathrm{Def}(\mathcal{M})$, $\widetilde{\mathrm{Def}}(\mathcal{M}) \equiv \widetilde{\mathrm{Def}}(\mathcal{M}^{\mathcal{U}})$.
- By the Keisler-Shelah theorem, $\mathcal{M} \equiv \mathcal{N} \Rightarrow (\exists \mathcal{U})\ \mathcal{M}^{\mathcal{U}} \cong \mathcal{N}(\mathcal{U})$.
- Thus, $\widetilde{\mathrm{Def}}(\mathcal{M}) \equiv_{\exists_1} \widetilde{\mathrm{Def}}(\mathcal{M}^{\mathcal{U}}) \cong \widetilde{\mathrm{Def}}(\mathcal{N}^{\mathcal{U}}) \equiv_{\exists_1} \widetilde{\mathrm{Def}}(\mathcal{N})$.

$\Diamond$

Some failures of invariance

It can happen that $\mathcal{M} \equiv \mathcal{N}$ but $K_0(\mathcal{M}) \not\equiv_{\forall\exists} K_0(\mathcal{N})$.

Example 3 Take $\mathcal{L} = \mathcal{L}(E)$ where E is a binary relation. Let $\mathcal{M}$ be the $\mathcal{L}$-structure on which E is an equivalence relation, $\mathcal{M}$ has one equivalence class of each finite cardinality, and $\mathcal{M}$ has no infinite classes. Let $\mathcal{N} \succ \mathcal{M}$ be a proper elementary extension. Set r :=the number of infinite equivalence classes in $\mathcal{N}$. Then $K_0(\mathcal{M}) \cong \mathbb{Z}[X]$ while $K_0(\mathcal{N}) \cong \mathbb{Z}[\{X_i\}_{i \leq r}]$. These rings are distinguished by an $\forall\exists$-sentence.

There are examples of $\mathcal{M} \prec \mathcal{N}$ with $K^s(\mathcal{M}) = 0$ and $K^s(\mathcal{N}) \neq 0$.

Question 4 If $\mathcal{M}$ admits a strong Euler characteristic, do all elementary extensions of $\mathcal{M}$ also admit a strong Euler characteristic?

Pigeon Hole Principles

Definition 13 *The structure $\mathcal{M}$ satisfies the* Pigeon Hole Principle*, written $\mathcal{M} \models \mathrm{PHP}$, if whenever $f : A \to A$ is a definable injective function of definable sets, then f is surjective. $\mathcal{M}$ satisfies the* Onto Pigeon Hole Principle*, written $\mathcal{M} \models \mathrm{onto-PHP}$, if there is no definable bijection $f : A \to A \setminus \{*\}$ in $\mathcal{M}$.*

Proposition 14 $\mathcal{M} \models \mathrm{onto-PHP} \Leftrightarrow K_0(\mathcal{M}) \neq 0$

The Grothendieck group of $\mathbb{Q}_p$

Question 5 (Luc Bélair) Is there a definable (in the language of rings) bijection between $\mathbb{Q}_p$ and $\mathbb{Q}_p^\times$?

Theorem 15 (Jean-Pierre Serre) *The Grothendieck ring of the category of p-adic analytic manifolds is isomorphic to $\mathbb{Z}/(p-1)\mathbb{Z}$.*

Theorem 16 (Raf Cluckers, Deirdre Haskell) $K_0(\mathbb{Q}_p) = 0$

Corollary 17 *There is a definable bijection $f : \mathbb{Q}_p^n \to \mathbb{Q}_p^n \setminus \{(0, \ldots, 0)\}$ for some n.*

In fact, the Cluckers-Haskell proof gives the stronger result that the Grothendieck *group* of $\mathbb{Q}_p$ is zero from which one can construct an explicit definable bijection between $\mathbb{Q}_p$ and $\mathbb{Q}_p^\times$.

Ordered Euler characteristics

Definition 18 *On a rig* $(R, +, \cdot, 0, 1)$ *define*
$x \leq y \;\Leftrightarrow\; (\exists z \in R)\; x + z = y$.

Definition 19 *A partially ordered Euler characteristic on* $\mathcal{M}$ *is an* $\mathcal{L}_{\text{ring}}(\leq)$*-morphism* $\chi : \widetilde{\text{Def}}(\mathcal{M}) \to (R, +, \cdot, \leq, 0, 1)$ *where* $\leq$ *is a partial order on* R *satisfying*

- $0 < 1$,
- $(\forall x, y, z)\; x \leq y \to x + z \leq y + z$, *and*
- $(\forall x, y, z)\; (z > 0 \;\&\; x \leq y) \to z \cdot x \leq z \cdot y$.

Proposition 20 $\mathcal{M} \models PHP \Leftrightarrow \mathcal{M}$ *admits an ordered Euler characteristic.*

Ax's Theorem

Theorem 21 (James Ax) *If* $f : \mathbb{C}^n \to \mathbb{C}^n$ *is an injective polynomial mapping, then* f *is surjective.*

Theorem 22 *Every algebraically closed field admits an ordered Euler characteristic.*

Euler characteristics on limits: ultraproducts

If $\mathcal{M} = \prod_{i \in I} \mathcal{M}_i/\mathcal{U}$ is an ultraproduct, then $\widetilde{\mathrm{Def}}(\mathcal{M})$ is a substructure of $\prod_{i \in I} \widetilde{\mathrm{Def}}(\mathcal{M}_i)/\mathcal{U}$.

If for each $i \in I$ we have an Euler characteristic $\chi_i : \widetilde{\mathrm{Def}}(\mathcal{M}_i) \to R_i$, then the ultaproduct $\chi/\mathcal{U}$ (defined on $\prod_{i \in I} \widetilde{\mathrm{Def}}(M_i)/\mathcal{U}$) gives an Euler characteristic on $\mathcal{M}$ with values in $\prod_{i \in I} R_i/\mathcal{U}$.

Example 6 If M is an ultraproduct of finite structures, then M admits an ordered Euler characteristic with values in a nonstandard extension of $\mathbb{Z}$.

Euler characteristics on limits: direct limits

Definition 23 *Let $\mathcal{M}$ be a structure and $A \subseteq \mathcal{M}$. Then A is* definably closed in $\mathcal{M}$ *if for each $\mathcal{L}_A$-definable function $f : \mathcal{M}^n \to \mathcal{M}$ one has $f(A^n) \subseteq A$.*

Definition 24 *If $(I, \leq)$ is a directed set, then the filter of cones on I is $\mathcal{C} := \{Y \subseteq I : (\exists a \in I)\ \{b \in I : b \geq a\} \subseteq Y\}$.*

Proposition 7 *If $\mathcal{M} = \varinjlim_{i \in I} \mathcal{M}_i$ is a direct limit of definably closed substructures, $\mathcal{M}$ admits quantifier elimination in $\mathcal{L}$, and for each $i \in I$ we have an Euler characteristic $\chi_i : \widetilde{\mathrm{Def}}(\mathcal{M}_i) \to R_i$; then there is an Euler characteristic $\chi : \widetilde{\mathrm{Def}}(\mathcal{M}) \to \prod_{i \in I} R_i/\mathcal{C}$ defined by $\chi([X]) = [\chi_i(\varphi(\mathcal{M}_i))]_{\mathcal{C}}$ where φ is a quantifier-free definition of X.*

Grothendieck ring-theoretic proof of Ax's theorem

- Algebraically closed fields eliminate quantifiers. (Claude Chevallay, Alfred Tarski)
- Each finite field considered as a subset of its algebraic closure is definably closed.
- The algebraic closure of a finite field is a direct limit of its finite subfields.
- For any nonprincipal ultrafilter $\mathcal{U}$ on the set of prime numbers, $\mathbb{C} \cong \prod \mathbb{F}_p^{\mathrm{alg}}/\mathcal{U}$.

Thus, $\mathbb{C}$ admits a nontrivial ordered Euler characteristic. Hence, $\mathbb{C} \models$ PHP and Ax's Theorem follows as as a special case.

Some quotients and subrings of $K_0(\mathbb{C})$

The above construction of an ordered Euler characteristic on $\mathbb{C}$ can be used to show that $K_0(\mathbb{C})$ is very large.

Theorem 25 *If L is an algebraically closed field and $\{E_i(L) : i \in I\}$ is a family of pairwise non-isogeneous elliptic curves over L, then $\{\chi_0([E_i(L)]) : i \in I\}$ is algebraically independent in $K_0(L)$. In particular, there is a ring embedding $\mathbb{Z}[\{X_i : i \in 2^{\aleph_0}\}] \hookrightarrow K_0(\mathbb{C})$.*

Since each element of $K_0(\mathbb{C})$ is represented by a variety, cohomology theories on the category of affine complex algebraic varieties can yield Euler characteristics on $\mathbb{C}$. For example, Hodge theory yields a surjective map $K_0(\mathbb{C}) \to \mathbb{Z}[X, Y]$.

Motivic integrals and $K_0(\mathbb{C})$

Set $\mathbb{L} := \chi_0([\mathbb{C}]) \in K_0(\mathbb{C})$.

Set $\mathcal{M}_{\mathrm{loc}} := K_0(\mathbb{C})[\mathbb{L}^{-1}]$.

Define a filtration on $\mathcal{M}_{\mathrm{loc}}$ by letting $F^m\mathcal{M}_{\mathrm{loc}}$ be the group generated by $\{\chi_0([S(\mathbb{C})])\mathbb{L}^{-i} : i - \dim S \geq m, S$ an irreducible variety $\}$.

Let $\widehat{\mathcal{M}}$ be the completion of $\mathcal{M}_{\mathrm{loc}}$ with respect to this filtration.

Given a (pure dimensional affine) variety $X \subseteq \mathbb{A}^n$ defined over $\mathbb{C}$, Denef, following Kontsevich, defines a measure $\mu_X : \mathrm{Def}^n(\mathbb{C}[[t]]) \to \widehat{\mathcal{M}}$.

While μ_X is countably additive, it does not respect definable isomorphisms so that it cannot be used to produce an Euler characteristic on $\mathbb{C}[[t]]$.

Definition 26 *If $f : X(\mathbb{C}[[t]]) \to \mathbb{Z}$ is a definable function and $A \subseteq X(\mathbb{C}[[t]])$ is a definable set, then the* motivic integral of f over A *is $\int_A \mathbb{L}^{-f} d\mu_X$ if this integral converges.*

Schanuel dimensions

Definition 27 *A dimension d on a structure $\mathcal{M}$ is a rig homomorphism $d : \widetilde{\mathrm{Def}}(\mathcal{M}) \to D$ satisfying $d(x+x) = d(x)$ universally.*

On any rig R one defines a partial quasi-order $\preceq$ by $x \preceq y \;\Leftrightarrow (\forall n \in \mathbb{Z}_+)\ (\exists z \in R)\ n \cdot x + z = y$. Define an equivalence relation $x \sim y \Leftrightarrow x \preceq y\ \&\ y \preceq x$.

Definition 28 *The Schanuel dimension of a structure $\mathcal{M}$ is the quotient map* $\dim : \widetilde{\mathrm{Def}}(\mathcal{M}) \to \widetilde{\mathrm{Def}}(\mathcal{M})/\!\sim\ =: \mathcal{D}(\mathcal{M})$.

Examples of dimensions

- $\mathcal{D}(\mathbb{R}) \cong (\{-\infty\} \cup \omega, \vee, +, -\infty, 0) \cong \mathcal{D}(\mathbb{Q}_p)$
- $\mathcal{D}(\mathbb{Z}) = \{[\varnothing]_\sim, [\{0\}]_\sim, [\mathbb{Z}]_\sim\}$
- If R is any global stability theoretic rank (Morley, Lascar, Shelah), then R is a dimension.
- Given a cardinal κ, define $\kappa^* := \{0, 1\} \cup \{\lambda : \aleph_0 \leq \lambda \leq \kappa\}$. For a structure $\mathcal{M}$ of cardinality κ, the function $d : \widetilde{\mathrm{Def}}(\mathcal{M}) \to (\kappa^*, \vee, \cdot)$ defined by $d([X]) = 1$ if $0 < \|X\| < \aleph_0$ and $d([X]) = \|X\|$ otherwise is a dimension.

Finite structures with dimension and measure

Definition 29 (Dugald Macpherson and Charles Steinhorn) *A class $\mathcal{C}$ of finite $\mathcal{L}$-structures is an asymptotic class with dimension and measure if for any $\mathcal{L}$-formula $\varphi(x, y_1, \ldots, y_m)$ there are*

- *real numbers B and C,*
- *a natural number N,*
- *real numbers $\mu_1, \ldots, \mu_N$, and*
- *formulas $\psi_0(y_1, \ldots, y_n), \ldots, \psi_N(y_1, \ldots, y_m)$*

such that for any $\mathcal{M} \in \mathcal{C}$ and $b \in \mathcal{M}^m$

- *$\mathcal{M} \models \bigvee_{i=1}^{N} \psi_i(b)$ and*
- *if $\mathcal{M} \models \psi_i(b)$ for $i > 0$ then $|\, \|\varphi(\mathcal{M}; b)\| - \mu_i \|\mathcal{M}\|| < C\sqrt{\|\mathcal{M}\|}$, and*
- *if $\mathcal{M} \models \psi_0(b)$, then $\|\varphi(M, b)\| < B$.*

Pseudofinite structures

Definition 30 *An infinite structure $\mathcal{M}$ is* strongly pseudofinite *if it is isomorphic to an ultraproduct of finite structures. An infinite structure $\mathcal{M}$ is* pseudofinite *if every sentence true in $\mathcal{M}$ is satisfied by some finite structure.*

If $\mathcal{M}$ is pseudofinite, then $K_0(\mathcal{M})$ embeds as an ordered subring of an elementary extension of $\mathbb{Z}$.

Moreover, if $\mathcal{M}$ is strongly pseudofinite, then χ_0 is a strong Euler characteristic. In fact, χ_0 satisfies the Lebesgue conditions.

Definition 31 *An ordered Euler characteristic $\chi : \widetilde{\mathrm{Def}}(\mathcal{M}) \to R$ satisfies the upper (resp. lower) Lebesgue condition if whenever $\pi : E \to B$ is a definable function and $f \in R$ with $\chi([\pi^{-1}\{b\}]) \geq f$ (resp. $\leq f$) for all $b \in B$, then $\chi([E]) \geq f \cdot \chi([B])$ (resp. $\leq f \cdot \chi([B])$).*

Questions about Euler characteristics on pseudofinite structures

Question 8 Does $\chi_0 : \widetilde{\mathrm{Def}}(\mathcal{M}) \to K_0(\mathcal{M})$ always satisfy the Lebesgue conditions for $\mathcal{M}$ a pseudofinite structure? Is χ_0 always strong on pseudofinite structures?

Question 9 If $\mathcal{M}$ is infinite and $\chi_0 : \widetilde{\mathrm{Def}}(\mathcal{M}) \to K_0(\mathcal{M})$ satisfies the Lebesgue conditions, must $\mathcal{M}$ be pseudofinite?

Fields with strong ordered Euler characteristics

Theorem 32 (James Ax) *A field K is pseudofinite if and only if*

- *K is perfect: if* $\mathrm{charK} = p > 0$*, then* $K \models (\forall x)(\exists y)\ y^p = x$,
- $\mathrm{Gal}(K^{alg}/K) \cong \widehat{\mathbb{Z}}$*: for each natural number n, K has exactly one separable extension of degree n and that extension is Galois over K, and*
- *K is pseudoalgebraically closed: for each absolutely irreducible polynomial $f(X, Y) \in K[X, Y]$ there is some $(a, b) \in K^2$ with* $f(a, b) = 0$.

Definition 33 *A field K is* quasifinite *if K is perfect and* $\mathrm{Gal}(K^{alg}/K) \cong \widehat{\mathbb{Z}}$.

Theorem 34 *If the field K admits a nontrivial strong ordered Euler characteristic, then K is quasifinite.*

Proof of quasifiniteness

- Perfection requires only an ordered Euler characteristic. If $K \neq K^p$, then $\chi([K]) = \chi([K^p]) < \chi([K])$.
- An ordered Euler characteristic χ gives a leading term function $\ell_\chi : K_0(R) \to L_\chi$ defined by $\ell_\chi(x) = \ell_\chi(y) \Leftrightarrow$ $(\forall n \in \omega)\ n|\chi(x) - \chi(y)| < \chi(x)\ \&\ n|\chi(x) - \chi(y)| < \chi(y)$.
- Reduce to the case of K infinite.
- Identify $\{f \in K[X] : \deg f = n \text{ and } f \text{ is monic}\}$ with K^n.
- $\ell_\chi([\{f \in K[X] : \deg f = n\ \&\ f \text{ is irreducible }\}]) = \frac{1}{n}\ell_\chi([K])^n$
- If $[L : K] \geq n$, then $\ell_\chi(\{f : K[x]/(f) \cong L\}) \geq \frac{1}{n}\ell_\chi([K])^n$.

Some questions

Question 10 Is there a combinatorially transparent condition equivalent to $K^s(\mathcal{M}) \neq 0$?

Question 11 Is $\mathrm{Th}_{\mathcal{L}_{\mathrm{ring}}}(K_0(G))$ an invariant of $\mathrm{Th}_{\mathcal{L}(+,0)}(G)$ for G an abelian group?

Question 12 Are there transparent (though non-trivial) conditions on dim which imply simplicity?

Reference

JAN KRAJÍČEK and THOMAS SCANLON, Combinatorics with definable sets: Euler characteristics and Grothendieck rings, *Bulletin of Symbolic Logic* **6**, no 3., September 2000, pages 311 – 330.