Section 9.6: Applications of definite integrals
Income streams

If at time t the rate of income generated by some enterprise is given by the value of the function $f(t)$, then the total income generated between time a and time b is $\int_a^b f(t)dt$.

Valuation of future payments

What value should one assign to an expected future payment?

Review of continuously compounded interest

Recall that if one has a principal of P which earns continuously compounded interest at a rate of r, then after t years, the investment would be worth Pe^{rt}.

Present value

The present value of a payment of \$$A$ made t years in the future is the amount P for which with a principal of \$$P$ dollars invested for t years with continuously compounded interest of rate r one would earn \$$A$.

From the formula for continuously compounded interest, we conclude that $Pe^{rt} = A$ so that $P = Ae^{-rt}$.

Valuation of an income stream

Suppose that some enterprise produces income at a steady rate of $\$A$ per year. Of course, this income stream over the next T years will produce $\$AT$, but how much is it worth in present dollars?

A solution

We may approximate the continous income stream as one that is paid in discrete increments.

Suppose that between now and T years from now N payments are made at uniform intervals. Then, the length of time between each payment is $\Delta = \frac{T}{N}$ years.

The i^{th} payment of $A\Delta$ is made at time $i\Delta$. As such, if we assume an interest rate of r, it has a present value of $(A\Delta)e^{i\Delta}$.

So, the sum of the present values is $\sum_{i=1}^{N} Ae^{-i\Delta r}\Delta$.

Solution, continued

The expression

$$\sum_{i=1}^{N} Ae^{-i\Delta r}\Delta$$

is the right-hand approximation to

$$\int_0^T Ae^{-rt}dt$$

Example

Assuming an interest rate of 5%, compute the present value of a constant income stream of $\$100,000$ per year for 10 years.

Solution

$$\int_0^{10} 100,000e^{-(0.05)t}dt = -2,000,000e^{-(0.05)t}|_{t=0}^{t=10}$$
$$= -2,000,000e^{-\frac{1}{2}} + 2,000,000$$
$$\approx 786,938.68$$

Valuation of fluctuating income streams

If the income stream varies as a function of time, so that at time t, $A(t)$ is the rate at which the payments are made, and the interest rate also varies (possibly) as a function of time, given by the function $r(t)$, then the present value of the income stream over the next T years is

$$\int_0^T A(t)e^{-tr(t)}dt$$

Example

Suppose the interest rate is constantly 5% and the income stream is given by the function $A(t) = 1000 + 50t$. What is the present value of this income stream over the next 10 years?

Solution

$$\int_0^{10} (1000 + 50t)e^{-t(0.05)} dt \quad = \quad 1000 \int_0^{10} e^{\frac{-t}{20}} dt + 50 \int_0^{10} te^{\frac{-t}{20}} dt$$

We compute that

$$\begin{aligned} 1000 \int_0^{10} e^{\frac{-t}{20}} dt &= -20,000e^{-\frac{t}{20}}|_0^{10} \\ &= -20,000e^{\frac{-1}{2}} + 20,000 \\ &\approx 7,869.38 \end{aligned}$$

Solution, continued

Integrating by parts, with $u = t$ and $dv = e^{\frac{-t}{20}}dt$, so that $du = dt$ and $v = -20e^{\frac{-t}{20}}$.

So,

$$\begin{aligned}\int te^{\frac{-t}{20}}dt &= -20te^{\frac{-t}{20}} + 20\int e^{\frac{-t}{20}}dt \\ &= -20te^{\frac{-t}{20}} - 400e^{\frac{-t}{20}} + C\end{aligned}$$

Thus, $50\int_0^{10} te^{\frac{-t}{20}} = -30,000e^{\frac{-1}{2}} + 20,000 \approx 1804.08$.

So, the total present value is $\approx \$1804.08 + \$7869.38 = \$9673.46$.