Math 202B Solutions

Assignment 7

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25. Let f be in $L^p(\lambda_N)$ and g in $L^{p'}(\lambda_N)$. Prove that $f*g$ is bounded and continuous.

    Proof: Clearly, for any x, $|(f*g)(x)| = \left|\int_{\mathbb{R}^N} f(y)g(x-y)\,d\lambda_N(y)\right| \le \|f\|_p\|g\|_{p'}$, so $f*g$ is bounded. Also, if T_a represents the operation of translation by a (i.e. $T_af(x) = f(x-a)$), then $T_a(f*g) = (T_af)*g$. Proof:

$$(f*g)(x-a) = \int_{\mathbb{R}^N} f(y)g(x-a-y)\,dy = \int_{\mathbb{R}^N} f(y-a)g(x-y)\,dy = ((T_af)*g)(x).$$

    Thus, if $1 \le p < \infty$, then $\|T_a(f*g) - f*g\|_\infty \le \|T_af - f\|_p\|g\|_{p'} \to 0$ as $a \to 0$, which implies that $f*g$ is uniformly continuous. If $p = \infty$, use symmetry to reverse the roles of f and g.

26. (a) Let the function $\psi : \mathbb{R} \to \mathbb{R}$ be of class C^∞ and have compact support. Let the function g be in $L^p(\lambda)$ $(1 \le p \le \infty)$. Prove that $\psi * g$ is of class C^∞, with $(\psi*g)^{(n)} = \psi^{(n)} * g$ for all positive integers n.

    Proof: It will be enough to show that $\psi*g$ is differentiable with derivative $\psi'*g$; the desired conclusion will then follow by induction.

    For $\delta \neq 0$ define the function $D_\delta\psi$ by

$$D_\delta\psi(x) = \frac{\psi(x+\delta) - \psi(x)}{\delta}.$$

    One easily verifies that

$$\frac{(\psi*g)(x+\delta) - (\psi*g)(x)}{\delta} = (D_\delta\psi)*g(x).$$

    We now have $D_\delta\psi \to \psi'$ pointwise; also, by the mean value theorem, $D_\delta\psi(x) = \psi'(c)$ for some c between x and $x+\delta$. However, since ψ' is continuous and has compact support, it is uniformly continuous. This implies that $\|D_\delta\psi - \psi'\|_\infty \to 0$ as $\delta \to 0$. Also, if ψ is supported on $[-R, R]$ and $|\delta| < 1$, then $D_\delta\psi$ is supported on $[-R-1, R+1]$. If we let $K = [-R-1, R+1]$, then $\|D_\delta\psi - \psi'\|_{p'} \le \|D_\delta\psi - \psi'\|_\infty m(K)^{1/p'} \to 0$ for $p' < \infty$ also.

    Therefore,

$$\left|\frac{(\psi*g)(x+\delta) - (\psi*g)(x)}{\delta} - (\psi'*g)(x)\right| \le \|D_\delta\psi - \psi'\|_{p'}\|g\|_p \to 0 \text{ as } \delta \to 0,$$

    which gives the desired conclusion.

    (b) Use Part (a) to prove that $C^\infty(\mathbb{R}) \cap L^p(\lambda)$ is dense in $L^p(\lambda)$ for $1 \le p < \infty$.

    Proof: Take a nonnegative C^∞ function ψ of compact support such that $\|\psi\|_1 = 1$. For example, let

$$\psi(x) = \begin{cases} C\exp\left(\frac{1}{x^2-1}\right), & |x| < 1 \\ 0, & |x| \ge 1, \end{cases}$$

    where the constant C is chosen so that $\|\psi\|_1 = 1$. For $\epsilon > 0$ let $\psi_\epsilon(x) = \frac{1}{\epsilon}\psi(\frac{x}{\epsilon})$.

    Let g be in $L^p(\lambda)$ $(1 \le p < \infty)$. By Theorem 15.1 in lecture we have $\psi_\epsilon * g \to g$ in L^p norm. By part (a), each function $\psi_\epsilon * g$ is of class C^∞, giving the desired conclusion.

27. Let E be a Lebesgue measurable subset of $\mathbb{R}^N$ of finite positive measure. Prove that the set

$$E - E = \{x - y : x, y \in E\}$$

    contains a neighborhood of the origin.

    Proof: Consider the convolution $\chi_E * \chi_{-E}$; we calculate

$$(\chi_E * \chi_{-E})(a) = \int_{\mathbb{R}^N} \chi_E(t)\chi_{-E}(a-t)\,d\lambda_N(t) = \int_{\mathbb{R}^N} \chi_E(t)\chi_{E+a}(t)\,d\lambda_N(t) = \lambda_N(E \cap (E+a)).$$

By problem 25, since $\chi_E \in L^1(\lambda_N)$ and $\chi_{-E} \in L^\infty(\lambda_N)$, this means that $\lambda_N(E \cap (E+a))$ is a continuous function of a. By hypothesis this function has value $\lambda_N(E) > 0$ at $a = 0$, so the function is positive for a in some neighborhood of 0. In particular, for any such a, $E \cap (E+a) \neq \emptyset$, so $a \in E - E$.

28. Prove that the algebra $L^1(\lambda)$ has no identity ($\lambda =$ Lebesgue measure on $\mathbb{R}$).

    Proof 1: Suppose, to the contrary, that there were a function $e \in L^1(\lambda)$ such that $e * g = g$ for each $g \in L^1(\lambda)$. Let $\psi = \chi_{[0,1]}$, and for $\epsilon > 0$ let $\psi_\epsilon(x) = \frac{1}{\epsilon}\psi(\frac{x}{\epsilon})$. Then by Theorem 15.1 in lecture, we have $\psi_\epsilon = e * \psi_\epsilon \to e$ in $L^1(\lambda)$ as $\epsilon \to 0$. However, this is impossible, since a straightforward calculation shows that $\|\psi_{2^{-n}} - \psi_{2^{-(n+1)}}\|_1 = 1$, so the sequence $(\psi_{2^{-n}})_{n=1}^\infty$ is not Cauchy in $L^1(\lambda)$.

    Proof 2: If $e \in L^1(\lambda)$ were an identity, then by problem 25 we would have $e * \chi_{[0,1]} = \chi_{[0,1]}$ is equal to some continuous function almost everywhere. However, this is impossible since $\chi_{[0,1]}$ clearly remains discontinuous at 0 and 1 after any modification on a set of measure zero.