Math 202B Solutions

Assignment 6

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21. Suppose the sequence $(f_n)_1^\infty$ in $L^1(\mu)$ converges almost everywhere to the function f in $L^1(\mu)$, and that $\lim_{n\to\infty}\|f_n\|_1 = \|f\|_1$. Prove that $\lim_{n\to\infty}\|f - f_n\|_1 = 0$.

Proof 1: For each n, let $E_n = \{|f_n| \le 2|f|\}$, and let $g_n = f_n\chi_{E_n}$. Then if $f(x) \neq 0$ and $f_n(x) \to f(x)$ as $n \to \infty$, then $x \in E_n$ for n sufficiently large, so $g_n(x) \to f(x)$. Also, if $f(x) = 0$, then $g_n(x) = 0$ for each n. Therefore, $g_n \to f$ almost everywhere.

Also, $|g_n| \le 2|f|$ by construction, so by the dominated convergence theorem, $\int |g_n|\,d\mu \to \int |f|\,d\mu = \|f\|_1$ as $n \to \infty$. On the other hand, by assumption, $\int |f_n|\,d\mu \to \int |f|\,d\mu$ as $n \to \infty$. Therefore, $\int(|f_n| - |g_n|)\,d\mu \to 0$ as $n \to \infty$. However, $|f_n| - |g_n| = |f_n|(1 - \chi_{E_n}) = |f_n - g_n|$, so we see that $\|f_n - g_n\|_1 \to 0$ as $n \to \infty$.

We now have

$$\|f - f_n\|_1 \le \|f - g_n\|_1 + \|f_n - g_n\|_1.$$

However, $|f - g_n| \le 3|f|$, so by the dominated convergence theorem, $\|f - g_n\|_1 \to 0$ as $n \to \infty$. Thus, $\|f - g_n\|_1 + \|f_n - g_n\|_1 \to 0$ as $n \to \infty$, so $\lim_{n\to\infty}\|f - f_n\|_1 = 0$ also, as desired.

Proof 2: Consider the sequence of nonnegative functions $g_n = |f| + |f_n| - |f - f_n|$. Then

$$\liminf_{n\to\infty} g_n = \lim_{n\to\infty}(|f| + |f_n| - |f - f_n|) = 2|f|,$$

while

$$\begin{aligned}\liminf_{n\to\infty}\int g_n\,d\mu &= \liminf_{n\to\infty}\left(\int |f|\,d\mu + \int |f_n|\,d\mu - \int |f - f_n|\,d\mu\right)\\ &= \int |f|\,d\mu + \lim_{n\to\infty}\int |f_n|\,d\mu - \limsup_{n\to\infty}\int |f - f_n|\,d\mu.\end{aligned}$$

Therefore, Fatou's Lemma gives

$$2\|f\|_1 = \int(\liminf_{n\to\infty} g_n)\,d\mu \le \liminf_{n\to\infty}\int g_n\,d\mu = 2\|f\|_1 - \limsup_{n\to\infty}\int |f - f_n|\,d\mu,$$

which implies $\limsup_{n\to\infty}\|f - f_n\|_1 \le 0$. Since obviously $\liminf_{n\to\infty}\|f - f_n\|_1 \ge 0$, this implies the desired result.

22. Let $(X, \mathcal{R}, \mu)$ and $(Y, \mathcal{S}, \nu)$ be σ-finite measure spaces. Let A be a set in the hereditary σ-ring generated by $\mathcal{R}$ and B a set in the hereditary σ-ring generated by $\mathcal{S}$. Prove that $(\mu \times \nu)^*(A \times B) = \mu^*(A)\nu^*(B)$.

Proof: If $R \in \mathcal{R}$ and $A \subseteq R$, then $\mu^*(A) \le \mu(R)$, and there is an $R \in \mathcal{R}$ for which equality holds. (See the solution to problem 5 for the proof of this last statement.) Similarly, if $S \in \mathcal{S}$ and $B \subseteq S$, then $\nu^*(B) \le \nu(S)$, and there is an S for which equality holds. And if $E \in \mathcal{R} \times \mathcal{S}$ and $A \times B \subseteq E$, then $\mu^*(A \times B) \le \mu \times \nu(E)$, and there is an E for which equality holds. (We can use $\mathcal{R} \times \mathcal{S}$ in the last statement, rather than the possibly larger σ-ring of $\mu \times \nu$-measurable sets, because every $\mu \times \nu$-measurable set differs from a set in $\mathcal{R} \times \mathcal{S}$ by a null set.)

Take $R \in \mathcal{R}$ such that $A \subseteq R$ and $\mu^*(A) = \mu(R)$, and $S \in \mathcal{S}$ such that $B \subseteq S$ and $\nu^*(B) = \nu(S)$. Let $E = R \times S$. Then $A \times B \subseteq E$, and

$$(\mu \times \nu)^*(A \times B) \le \mu \times \nu(R \times S) = \mu(R)\nu(S) = \mu^*(A)\nu^*(B).$$

It only remains to prove the reverse inequality.

Let $E \in \mathcal{R} \times \mathcal{S}$ with $A \times B \subseteq E$, and $\mu \times \nu(E) = (\mu \times \nu)^*(A \times B)$. Then the function $y \mapsto \mu(E^y)$ is $\mathcal{S}$-measurable. Thus, letting $F = \{y \in Y : \mu(E^y) \ge \mu^*(A)\}$, we have $F \in \mathcal{S}$, and $B \subseteq F$ since $A \subseteq E^y$ whenever $y \in B$. Therefore,

$$(\mu \times \nu)^*(A \times B) = \mu \times \nu(E) = \int \mu(E^y)\,d\nu(y) \ge \int_F \mu(E^y)\,d\nu(y) \ge \mu^*(A)\nu(F) \ge \mu^*(A)\nu^*(B).$$

23. (a) Prove that if f is in $L^1(\mu)$ then $\lim_{t\to\infty} t\Lambda_t(f) = 0$.

Proof: Let $E_t = \{|f| > t\}$, so that $\mu(E_t) = \Lambda_t(f)$. Then $t\Lambda_t(f) = \int_{E_t} t\,d\mu \le \int_{E_t} |f|\,d\mu$. However, $\int_{E_t} |f|\,d\mu \to 0$ as $t \to \infty$ by the dominated convergence theorem, since $\bigcap E_t = \{|f| = \infty\}$ is a null set. Therefore, $t\Lambda_t(f) \to 0$ as $t \to \infty$ also.

(b) Prove that if μ is a finite measure, if f is a measurable function, and if there is a positive number ϵ such that $\Lambda_f(t) = O(1/t(\log t)^{1+\epsilon})$ as $t \to \infty$, then f is in $L^1(\mu)$.

Proof: We have

$$\begin{aligned}\int |f|\,d\mu = \int_0^\infty \Lambda_f(t)\,dt &= \int_0^a \Lambda_f(t)\,dt + \int_a^\infty \Lambda_f(t)\,dt \\ &\le a\mu(X) + \int_a^\infty \Lambda_f(t)\,dt.\end{aligned}$$

By assumption, there are positive numbers a, c such that $\Lambda_f(t) \le c/t(\log t)^{1+\epsilon}$ for $t \ge a$. (Without loss of generality assume $a > 1$.) Hence

$$\int |f|\,d\mu \le a\mu(X) + \int_a^\infty \frac{dt}{t(\log t)^{1+\epsilon}} = a\mu(X) + \int_{\log a}^\infty \frac{ds}{s^{1+\epsilon}} < \infty.$$

24. Let f be in $L^1(\lambda)$. Define the function g on $\mathbb{R}$ by

$$g(x) = \int_{\mathbb{R}} \frac{f(y)}{1 + x^2 + y^2}\,dy.$$

Prove that g is in $L^1(\lambda)$, and that $\|g\|_1 \le \pi\|f\|_1$.

Proof: It is easy to see that the function $\mathbb{R}^2 \to \mathbb{R}$ defined by $(x, y) \mapsto f(y)$ is Lebesgue measurable. In addition, $\frac{1}{1+x^2+y^2}$ is continuous and thus Borel measurable, so their product $\frac{f(y)}{1+x^2+y^2}$ is Lebesgue measurable. Thus, the fiberwise integral g is measurable, and we have

$$\|g\|_1 \le \int_{\mathbb{R}} \left(\int_{\mathbb{R}} \frac{|f(y)|}{1 + x^2 + y^2}\,dy \right) dx = \int_{\mathbb{R}} \left(|f(y)| \int_{\mathbb{R}} \frac{1}{1 + x^2 + y^2}\,dx \right) dy.$$

Since $\int_{-\infty}^{\infty} \frac{1}{1+x^2+y^2}\,dx = \frac{\pi}{\sqrt{1+y^2}} \le \pi$, we get $\|g\|_1 \le \pi \int_{\mathbb{R}} |f(y)|\,dy = \pi\|f\|_1$.