Math 202B Solutions

Assignment 5
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17. Let f and g be in $L^p(\mu)$, where $1 < p < \infty$. Prove that the inequality $\|f + g\|_p \le \|f\|_p + \|g\|_p$ is strict unless $f = 0$ a.e., or $g = 0$ a.e., or g is a positive scalar multiple of f.

Proof: The only nontrivial part is to show that if $\|f\|_p\|g\|_p \neq 0$ and $\|f + g\|_p = \|f\|_p + \|g\|_p$, then g is a positive scalar multiple of f. This will be done by reviewing the proof of Minkowski's inequality and determining what must be true if each of the inequalities in it is actually an equality. Since the proof of Minkowski's inequality used Hölder's inequality, we first need the condition for Hölder's inequality to be an equality.

Lemma 1. *Let $h \in L^p(\mu)$ and $h' \in L^{p'}(\mu)$. Then $\|hh'\|_1 = \|h\|_p\|h'\|_{p'}$ if and only if $\|h\|_p\|h'\|_{p'} = 0$ or $\frac{|h|^p}{\|h\|_p^p} = \frac{|h'|^{p'}}{\|h'\|_{p'}^{p'}}$ a.e.*

Proof. The case $\|h\|_p\|h'\|_{p'} = 0$ is trivial, so we assume $\|h\|_p\|h'\|_{p'} \neq 0$. Recall that in the proof of Hölder's inequality we used the inequality $a^{1-t}b^t \le (1-t)a + tb$ $(a, b > 0,\ 0 < t < 1)$, which was deduced from the concavity of the logarithm function. Since the logarithm function is strictly concave, equality holds if and only if $a = b$. Now, in the preceding inequality we set $a = |h|^p/\|h\|_p^p$, $b = |h'|^{p'}/\|h'\|_{p'}^{p'}$, $t = \frac{1}{p'}$ to get

$$\frac{|h|\,|h'|}{\|h\|_p\|h'\|_{p'}} \le \frac{1}{p}\left(\frac{|h|^p}{\|h\|_p^p}\right) + \frac{1}{p'}\left(\frac{|h'|^{p'}}{\|h'\|_{p'}^{p'}}\right).$$

Integration of this gives Hölder's inequality. Thus, Hölder's inequality is an equality if and only if the preceding inequality is an equality almost everywhere. By the above comments, this happens if and only if $a = b$ a.e., so the desired equivalence holds. □

Returning now to Minkowski's inequality, assume $\|f + g\|_p = \|f\|_p + \|g\|_p$. We then have

$$\|f\|_p + \|g\|_p = \|f + g\|_p \le \||f| + |g|\|_p \le \|f\|_p + \|g\|_p,$$

so that $\|f+g\|_p = \||f|+|g|\|_p$. The pointwise inequality $|f+g|^p \le (|f|+|g|)^p$ must therefore be an equality almost everywhere. Thus $|f + g| = |f| + |g|$ almost everywhere.

In the proof of Minkowski's inequality, the first step was to note the inequality

$$|f + g|^p \le |f||f + g|^{p-1} + |g||f + g|^{p-1},$$

which under the present assumptions reduces to an equality, at least almost everywhere. The second step was to apply Hölder's inequality to each of the products on the right side. In each case Hölder's inequality must reduce to an equality. Applying the condition for equality in Hölder's inequality, we conclude that

$$\frac{|f|^p}{\|f\|_p^p} = \frac{|f + g|^p}{\|f + g\|_p^p} \text{ a.e.}$$
$$\frac{|g|^p}{\|g\|_p^p} = \frac{|f + g|^p}{\|f + g\|_p^p} \text{ a.e.}$$

Hence $|g| = \frac{\|g\|_p}{\|f\|_p}|f|$ a.e., which together with the equality $|f + g| = |f| + |g|$ a.e., implies that $g = cf$ with $c = \frac{\|g\|_p}{\|f\|_p}$.

18. Let $1 \le p_0 < p_1 < \infty$. For $0 < t < 1$ define p_t by

$$\frac{1}{p_t} = \frac{1-t}{p_0} + \frac{t}{p_1}.$$

Prove that if f is in $L^{p_0}(\mu) \cap L^{p_1}(\mu)$, then f is in $L^{p_t}(\mu)$ and

$$\|f\|_{p_t} \le \|f\|_{p_0}^{1-t} \|f\|_{p_1}^{t}.$$

Proof: We have $\frac{(1-t)p_t}{p_0} + \frac{tp_t}{p_1} = 1$, so the numbers $q = \frac{p_0}{(1-t)p_t}$ and $q' = \frac{p_1}{tp_t}$ are conjugate indices. By Hölder's inequality,

$$\begin{aligned}
\|f\|_{p_t}^{p_t} &= \int |f|^{p_t}\, d\mu = \int |f|^{(1-t)p_t} |f|^{tp_t}\, d\mu \\
&\le \||f|^{(1-t)p_t}\|_q \||f|^{tp_t}\|_{q'} \\
&= \left(\int |f|^{p_0}\, d\mu\right)^{(1-t)p_t/p_0} \left(\int |f|^{p_1}\, d\mu\right)^{tp_t/p_1} \\
&= \|f\|_{p_0}^{(1-t)p_t} \|f\|_{p_1}^{tp_t},
\end{aligned}$$

and the desired inequality follows.

19. Let f be in $L^\infty(\mu)$ and in $L^p(\mu)$ for some finite p. Prove that $\|f\|_\infty = \lim_{p\to\infty} \|f\|_p$.

    Proof: Fix some finite p_0 such that $f \in L^{p_0}(\mu)$. For $p > p_0$ we have $\int |f|^p\, d\mu \le \|f\|_\infty^{p-p_0} \int |f|^{p_0}\, d\mu$, so $\|f\|_p \le \|f\|_\infty^{1-p_0/p} \|f\|_{p_0}^{p_0/p}$. Letting $p \to \infty$, we obtain $\limsup_{p\to\infty} \|f\|_p \le \|f\|_\infty$.

    Let ϵ be in $(0, \|f\|_\infty)$, and let $E_\epsilon = \{|f| > \|f\|_\infty - \epsilon\}$. Then E_ϵ is a set of finite positive measure, and $\int |f|^p\, d\mu \ge (\|f\|_\infty - \epsilon)^p \mu(E_\epsilon)$, giving $\|f\|_p \ge (\|f\|_\infty - \epsilon)\mu(E_\epsilon)^{1/p}$. Letting $p \to \infty$, we obtain $\liminf_{p\to\infty} \|f\|_p \ge \|f\|_\infty - \epsilon$. Since ϵ was arbitrary, we can conclude that $\liminf_{p\to\infty} \|f\|_p \ge \|f\|_\infty$.

    The desired equality has been established.

20. Let λ be Lebesgue measure on $\mathbb{R}$. Let p be in $(1, \infty)$. Construct a function that is in $L^p(\lambda)$ but that fails to be in $L^q(\lambda)$ for q in $[1, \infty) \setminus \{p\}$.

    Solution 1: Step 1: Let $r \in (p, \infty)$. Then there is a function in $L^p(\lambda)$ that is not in $L^q(\lambda)$ for $r \le q \le \infty$. (Example: $t^{-1/r}\chi_{(0,1)}$.)

    Step 2: Let $s \in (1, p)$. Then there is a function in $L^p(\lambda)$ that is not in $L^q(\lambda)$ for $1 \le q \le s$. (Example: $t^{-1/s}\chi_{[1,\infty)}$.)

    Step 3: Choose a sequence $(r_n)_1^\infty \subset (p, \infty)$ converging to p and a sequence $(s_n)_1^\infty \subset (1, p)$ converging to p. By step 1, for each n there is a nonnegative function $g_n \in L^p(\lambda)$ such that $\|g_n\|_p = 2^{-n}$ but such that $g_n \notin L^q(\lambda)$ for $q \ge r_n$. Also, by step 2, for each n there is a nonnegative function $h_n \in L^p(\lambda)$ such that $\|h_n\|_p = 2^{-n}$ but such that $h_n \notin L^q(\lambda)$ for $1 \le q \le s_n$. Let

    $$f = \sum_{n=1}^\infty g_n + \sum_{n=1}^\infty h_n.$$

    Then $f \in L^p(\lambda)$ since $\sum_{n=1}^\infty \|g_n\|_p + \sum_{n=1}^\infty \|h_n\|_p < \infty$. However, for $q > p$, $q \ge r_n$ for some n, so $g_n \notin L^q(\lambda)$, which implies $f \notin L^q(\lambda)$ since $f \ge g_n$. Similarly, for $q < p$, $q \le s_n$ for some n, so $f \ge h_n \notin L^q(\lambda)$.

    Solution 2: Let $f(x) = x^{-1/p}(\log x)^{-2/p}$ if $0 < x < \frac{1}{2}$ or $x > 2$, and 0 otherwise. Then a straightforward calculation shows that $\int f^q\, d\lambda$ diverges at ∞ if $q < p$, and $\int f^q\, d\lambda$ diverges at 0 if $q > p$, but $f \in L^p(\lambda)$.