Math 202B Solutions

Assignment 4

D. Sarason

13. Let $(X, \mathcal{A}, \mu)$ be a finite measure space and let f be a nonnegative measurable function on X. Prove that f is integrable if and only if

$$\sum_{n=1}^{\infty} \mu(\{f > n\}) < \infty.$$

Proof: Let $E_n = \{f > n\}$, and consider the function $g = \sum_{n=1}^{\infty} \chi_{E_n}$. Then it is easy to see that $g = n$ on $E_n \setminus E_{n+1} = \{n < f \leq n+1\}$, while $g = 0$ if $f \leq 1$; thus, we see that $g \leq f \leq g + 1$. Therefore,

$$\int g\, d\mu \leq \int f\, d\mu \leq \int g\, d\mu + \mu(X).$$

Since $\int g\, d\mu = \sum_{n=1}^{\infty} \mu(\{f > n\})$ by an easy application of the monotone convergence theorem, this implies that $\int f\, d\mu$ is finite if and only if $\sum_{n=1}^{\infty} \mu(\{f > n\})$ is.

14. For α a real number, define the function f_α on $\mathbb{R}$ by $f_\alpha(x) = |x|^{2\alpha}/(1 + x^2)$. Prove that f is Lebesgue integrable if and only if $-\frac{1}{2} < \alpha < \frac{1}{2}$.

Proof: Since f is an even function, it will suffice to consider its integrability on $(0, \infty)$, and we can treat separately the two subintervals $(0, 1]$ and $[1, \infty)$.

For $1 \leq x < \infty$ we have $\frac{1}{2}x^{2\alpha-2} \leq f_\alpha(x) \leq x^{2\alpha-2}$. Hence f_α is Lebesgue integrable on $[1, \infty)$ if and only if the function $x \mapsto x^{2\alpha-2}$ is, which happens if and only if the improper Riemann integral converges since this function is nonnegative. By elementary calculus, this is equivalent to $2\alpha - 2 < -1$, or $\alpha < \frac{1}{2}$. Similarly, for $0 < x \leq 1$ we have $\frac{1}{2}x^{2\alpha} \leq f_\alpha(x) \leq x^{2\alpha}$, so f is integrable on $(0, 1]$ if and only if the function $x \mapsto x^{2\alpha}$ is, which occurs if and only if $\alpha > -\frac{1}{2}$.

(In fact, it's easy to see using complex variables that $\int f_\alpha\, d\lambda = \pi/\cos(\pi\alpha)$ for $-\frac{1}{2} < \alpha < \frac{1}{2}$.)

15. Let f be a Lebesgue-integrable function on $\mathbb{R}$. Prove that the series

$$\sum_{n=-\infty}^{\infty} f(x + n)$$

converges absolutely for almost every x in $\mathbb{R}$.

Proof: Define $g : \mathbb{R} \to [0, \infty]$ by $g(x) = \sum_{n=-\infty}^{\infty} |f(x+n)|$; we need to prove that $g < \infty$ almost everywhere. However, since g is obviously periodic with period 1, it suffices to show that $g(x) < \infty$ for almost every $x \in [0, 1)$.

To see this, define $g_m : \mathbb{R} \to [0, \infty]$ by $g_m(x) = \sum_{n=-m}^{m} |f(x + n)|$. Then by the monotone convergence theorem $\int_{[0,1)} g_m\, d\lambda \to \int_{[0,1)} g\, d\lambda$ as $m \to \infty$. But

$$\int_{[0,1)} g_m\, d\lambda = \sum_{n=-m}^{m} \int_{[0,1)} |f(x + n)|\, d\lambda(x) = \sum_{n=-m}^{m} \int_{[n,n+1)} |f(x)|\, d\lambda(x) = \int |f|\chi_{[-m,m+1)}\, d\lambda.$$

Another application of the monotone convergence theorem shows that $\int |f|\chi_{[-m,m+1)}\, d\lambda \to \int |f|\, d\lambda < \infty$ as $m \to \infty$, so $\int_{[0,1)} g\, d\lambda < \infty$. This implies that $\lambda([0, 1) \cap \{g = \infty\}) = 0$, completing the proof.

16. Let f be a Lebesgue-integrable function on $\mathbb{R}^N$. For $r \geq 0$ let $B_r = \{x \in \mathbb{R}^N : \|x\| \leq r\}$, and define the function $g : [0, \infty) \to \mathbb{R}$ by

$$g(r) = \int_{B_r} f\, d\lambda_N$$

(λ_N = Lebesgue measure). Prove g is continuous.

Proof: Let $r_n \to r$ be any convergent sequence of nonnegative real numbers. Then clearly $\chi_{B_{r_n}} \to \chi_{B_r}$ everywhere except possibly on $\partial B_r = \{x : \|x\| = r\}$. Therefore, assuming that ∂B_r is Lebesgue null, we have $f\chi_{B_{r_n}} \to f\chi_{B_r}$ almost everywhere, and $|f\chi_{B_{r_n}}| \leq |f|$ with $f \in L^1(\lambda_N)$. Thus, by the dominated convergence theorem, $g(r_n) = \int f\chi_{B_{r_n}}\, d\lambda_N \to \int f\chi_{B_r}\, d\lambda_N = g(r)$, showing that g is continuous.

To show ∂B_r is Lebesgue null, we have for any $\epsilon > 0$ that $\partial B_r \subseteq B_r \setminus B_{r-\epsilon}$, so $\lambda_N(\partial B_r) \leq \lambda_N(B_r) - \lambda_N(B_{r-\epsilon}) = (r^N - (r-\epsilon)^N)\lambda_N(B_1)$. Since the last term approaches 0 as $\epsilon \to 0$, this shows that $\lambda_N(\partial B_r) = 0$. (Note: $\lambda_N(B_1) = \pi^{N/2}/\Gamma(\frac{N}{2}+1)$.)