Math 202B Solutions

Assignment 2

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5. Let μ be a finite measure on an algebra $\mathcal{A}$ of subsets of a set X. Prove that a subset E of X is μ^*-measurable if and only if $\mu^*(E) + \mu^*(X \setminus E) = \mu(X)$. (The finiteness assumption on μ is just the assumption that $\mu(X) < \infty$.)

   Proof: Since $\mu^*(X) = \mu(X)$, the "only if" part is obvious. For the other direction, assume $\mu^*(E) + \mu^*(X \setminus E) = \mu(X)$. We claim that there is some $G \in \mathcal{A}_{\sigma\delta}$ such that $G \supseteq E$ and $\mu^*(G) = \mu^*(E)$. To construct such a G, first note that for each $\epsilon > 0$, there is a countable cover of E by sets in $\mathcal{A}$ whose outer measures sum to less that $\mu^*(E) + \epsilon$. Thus, for each $n > 0$ choose such a cover $(A_{in})_{i=1}^{\infty}$ for $\epsilon = \frac{1}{n}$, and let $G_n = \bigcup_{i=1}^{\infty} A_{in} \in \mathcal{A}_\sigma$, so that $\mu^*(G_n) \leq \sum_{i=1}^{\infty} \mu^*(A_{in}) < \mu^*(E) + \frac{1}{n}$. Now if we let $G = \bigcap_{n=1}^{\infty} G_n \in \mathcal{A}_{\sigma\delta}$, then $E \subseteq G$, and for each n, $\mu^*(E) \leq \mu^*(G) \leq \mu^*(G_n) < \mu^*(E) + \frac{1}{n}$, so $\mu^*(G) = \mu^*(E)$ as desired.

   But now G is μ^*-measurable; thus, by setting $A = X \setminus E$ in $\mu^*(A) = \mu^*(A \cap G) + \mu^*(A \setminus G)$, we get $\mu^*(X \setminus E) = \mu^*(G \setminus E) + \mu^*(X \setminus G)$. Adding $\mu^*(E) = \mu^*(G)$ to both sides gives $\mu(X) = \mu^*(G \setminus E) + \mu(X)$, so $\mu^*(G \setminus E) = 0$ since $\mu(X) < \infty$. Therefore, $G \setminus E$ is μ^*-measurable, so $E = G \setminus (G \setminus E)$ is also μ^*-measurable.

6. Let μ be a measure on a ring $\mathcal{R}$ and let E be a set in the hereditary σ-ring generated by $\mathcal{R}$. Prove that for E to be μ^*-measurable it is sufficient that

   $$\mu(A) = \mu^*(A \cap E) + \mu^*(A \setminus E)$$

   for every A in $\mathcal{R}$ such that $\mu(A) < \infty$.

   Proof 1: Assume E satisfies the given condition, and consider first the case where there is a set $A \in \mathcal{R}$ with $\mu(A) < \infty$ and $E \subseteq A$. Let $\mathcal{R}_A$ be the algebra $\mathcal{R} \cap P(A)$ on A (i.e. the sets in $\mathcal{R}$ which are contained in A), and let $\mu_A = \mu|\mathcal{R}_A$. One easily checks that $\mu_A^* = \mu^*|P(A)$. By problem 5, the set E is μ_A^*-measurable. Consider now any set S in the hereditary σ-ring generated by $\mathcal{R}$. Because A itself is μ^*-measurable,

   $$\mu^*(S) = \mu^*(S \cap A) + \mu^*(S \setminus A),$$

   and because E is μ_A^*-measurable,

   $$\mu^*(S \cap A) = \mu^*((S \cap A) \setminus E) + \mu^*(S \cap E).$$

   Hence

   $$\mu^*(S) = \mu^*(S \setminus A) + \mu^*((S \cap A) \setminus E) + \mu^*(S \cap E) \geq \mu^*(S \setminus E) + \mu^*(S \cap E),$$

   which establishes the μ^*-measurability of E.

   Consider now the general case. Let S be any set in the hereditary σ-ring. We want to show that $\mu^*(S) \geq \mu^*(S \cap E) + \mu^*(S \setminus E)$, so we may assume $\mu^*(S) < \infty$. Then there is a sequence $A_1, A_2, \ldots$ of sets in $\mathcal{R}$ that covers S and satisfies $\sum_{n=1}^{\infty} \mu(A_n) < \infty$. In particular, $\mu(A_n) < \infty$ for every n. By the first part of the proof, each set $E \cap A_n$ is μ^*-measurable, from which we conclude $\bigcup_{n=1}^{\infty}(E \cap A_n) = E \cap \bigcup_{n=1}^{\infty} A_n$ is μ^*-measurable. Because $S \subseteq \bigcup_{n=1}^{\infty} A_n$, the desired inequality follows.

   Proof 2: First note that the restriction $\mu(A) < \infty$ is superfluous, since if $\mu(A) = \infty$ then the inequality $\mu(A) = \mu^*(A) \leq \mu^*(A \cap E) + \mu^*(A \setminus E)$ automatically becomes an equality. Now for any A in the hereditary σ-ring generated by $\mathcal{R}$, suppose we have $B_n \in \mathcal{R}$ such that $A \subseteq \bigcup_{n=1}^{\infty} B_n$. Then $A \cap E \subseteq \bigcup_{n=1}^{\infty}(B_n \cap E)$, and $A \setminus E \subseteq \bigcup_{n=1}^{\infty}(B_n \setminus E)$. Therefore,

   $$\sum_{n=1}^{\infty} \mu(B_n) = \sum_{n=1}^{\infty} \mu^*(B_n \cap E) + \sum_{n=1}^{\infty} \mu^*(B_n \setminus E) \geq \mu^*(A \cap E) + \mu^*(A \setminus E).$$

   Taking the infimum over all such covers $\{B_n\}$, this implies that $\mu^*(A) \geq \mu^*(A \cap E) + \mu^*(A \setminus E)$.

7. Construct a meager subset of $\mathbb{R}^N$ whose complement has Lebesgue measure 0.

Solution 1: Step 1: For each $\epsilon > 0$, we construct an open dense set $E \subseteq \mathbb{R}^N$ such that $\lambda_N(E) < \epsilon$. To do this, choose a countable basis $\{U_n\}$ for $\mathbb{R}^N$. Now for each n, let E_n be a nonempty open subset of U_n such that $\lambda_N(E_n) < \epsilon/2^n$. Then setting $E = \bigcup_{n=1}^\infty E_n$, $\lambda_N(E) \leq \sum_{n=1}^\infty \lambda_N(E_n) < \epsilon$, and E is open and dense by construction.

Step 2: For each integer $n > 0$, use step 1 to choose an open dense set $E_n \subseteq \mathbb{R}^N$ with $\lambda_N(E_n) < \frac{1}{n}$, and set $E = \bigcap_{n=1}^\infty E_n$. Then E is residual, and for each n, $\lambda_N(E) \leq \lambda_N(E_n) < \frac{1}{n}$, so $\lambda_N(E) = 0$. Thus, $\mathbb{R}^N \setminus E$ is a meager set whose complement has Lebesgue measure 0.

Solution 2: Step 1: For each $\alpha \in [0, 1)$, we construct a nowhere dense subset of $[0, 1]$ with Lebesgue measure α. We do this by generalizing the construction of the Cantor set: let μ_n be a strictly decreasing sequence such that $\mu_0 = 1$ and $\mu_n \to \alpha$. Now set $X_0 = [0, 1]$, and for $n \geq 1$ construct X_n by removing the middle open interval of length $\frac{\mu_{n-1}-\mu_n}{2^{n-1}}$ from each interval of X_{n-1}. Thus, X_n consists of 2^n closed intervals, each of length $\mu_n/2^n$. Set $X = \bigcap_{n=0}^\infty X_n$; then X is closed and has empty interior (since the lengths of intervals in X_n get arbitrarily small), so X is nowhere dense. Also, since $\lambda(X_0) < \infty$, $\lambda(X) = \lim_{n\to\infty} \lambda(X_n) = \lim_{n\to\infty} \mu_n = \alpha$.

Step 2: We construct a meager set $X \subseteq [0, 1]$ with $\lambda(X) = 1$, which implies that $[0, 1] \setminus X$ has Lebesgue measure 0. To do this, for each n, use step 1 to construct a nowhere dense $X_n \subseteq [0, 1]$ with $\lambda(X_n) = 1 - \frac{1}{n}$. Set $X = \bigcup_{n=1}^\infty X_n$; then X is meager by construction, and $1 \geq \lambda(X) \geq \lambda(X_n) = 1 - \frac{1}{n}$ for each n, so $\lambda(X) = 1$.

Step 3: We construct a meager set $X \subseteq \mathbb{R}$ such that $\mathbb{R} \setminus X$ has Lebesgue measure 0 (thus we will be done for $N = 1$). To do this, use step 2 to find meager $Y \subseteq [0, 1]$ with $\lambda(Y) = 1$, and set $X = \bigcup_{n=-\infty}^\infty (Y + n)$, where $Y + n = \{y + n : y \in Y\}$ is the translate of Y to the right by n. Then X is again meager, and $\mathbb{R} \setminus X = \bigcup_{n=-\infty}^\infty ([n, n + 1] \setminus (Y + n))$, where each term is Lebesgue null by translation invariance, so $\mathbb{R} \setminus X$ is also Lebesgue null.

Step 4: For arbitrary N, set $X_N = X \times \mathbb{R}^{N-1}$. It is easy to see that X_N is meager in $\mathbb{R}^N$, and $\mathbb{R}^N \setminus X$ is Lebesgue-null.

8. For $S \subseteq \mathbb{R}$, define $\mu^*(S)$ to be 0 if S is countable, 1 if S is uncountable but there is a countable set C such that $S \setminus C$ is bounded, and ∞ otherwise. Prove that μ^* is an outer measure, and describe the μ^*-measurable sets. Is every μ^*-measurable set σ-finite? Can μ^* be induced by a measure on a ring?

Solution: First, to prove that μ^* is an outer measure, it is obvious that $\mu^*(\emptyset) = 0$. Now suppose $A \subseteq B \subseteq \mathbb{R}$; if $\mu^*(B) = 0$, then B is countable, so A is countable also, and $\mu^*(A) = 0$. If $\mu^*(B) = 1$, choose a countable set C such that $B \setminus C$ is bounded. Now either A is countable, in which case $\mu^*(A) = 0$, or A is uncountable, in which case $A \setminus C \subseteq B \setminus C$ is bounded, so $\mu^*(A) = 1$. Finally, if $\mu^*(B) = \infty$, obviously $\mu^*(A) \leq \mu^*(B)$.

Now for countable subadditivity, suppose $A_1, A_2, \ldots \subseteq \mathbb{R}$; then we want to prove $\mu^*(\bigcup_{n=1}^\infty A_n) \leq \sum_{n=1}^\infty \mu^*(A_n)$. If $\sum_{n=1}^\infty \mu^*(A_n) = \infty$, the inequality is obvious; otherwise, we must have $\mu^*(A_n) < \infty$ for each n, and $\mu^*(A_n) = 1$ for only finitely many n. If $\mu^*(A_n) = 0$ for each n, then each A_n is countable, so $\bigcup_{n=1}^\infty A_n$ is countable, and $\mu^*(\bigcup_{n=1}^\infty A_n) = 0 = \sum_{n=1}^\infty \mu^*(A_n)$. Otherwise, by reordering we may assume $\mu^*(A_1) = \cdots = \mu^*(A_k) = 1$ and $\mu^*(A_n) = 0$ for $n > k$, for some k. Now choose C_n countable for $n \leq k$ such that $A_n \setminus C_n$ is bounded. Then $\bigcup_{n=1}^\infty A_n$ is uncountable since A_1 is, and

$$\left(\bigcup_{n=1}^\infty A_n\right) \setminus \left[\left(\bigcup_{n=1}^k C_n\right) \cup \left(\bigcup_{n=k+1}^\infty A_n\right)\right] \subseteq \bigcup_{n=1}^k (A_n \setminus C_n)$$

is bounded, while $(\bigcup_{n=1}^k C_n) \cup (\bigcup_{n=k+1}^\infty A_n)$ is countable. Therefore, $\mu^*(\bigcup_{n=1}^\infty A_n) = 1 \leq k = \sum_{n=1}^\infty \mu^*(A_n)$.

Now every countable set $C \subseteq \mathbb{R}$ is μ^*-measurable, since $\mu^*(C) = 0$; also, $\mathbb{R}$ is μ^*-measurable, since obviously $\mu^*(A) = \mu^*(A) + \mu^*(\emptyset) = \mu^*(A \cap \mathbb{R}) + \mu^*(A \setminus \mathbb{R})$ for every $A \subseteq \mathbb{R}$. Thus, cocountable sets (sets whose complements are countable) are also μ^*-measurable. We claim that, in fact, every μ^*-measurable set is either countable or cocountable. Thus, suppose $S \subseteq \mathbb{R}$ is μ^*-measurable. Then either $[-1, 1] \cap S$ or $[-1, 1] \setminus S$ is uncountable. If $[-1, 1] \setminus S$ is uncountable, then $[-N, N] \setminus S$ is uncountable for every positive integer N, so $\mu^*([-N, N] \setminus S) = 1$. Since $1 = \mu^*([-N, N]) = \mu^*([-N, N] \cap S) + \mu^*([-N, N] \setminus S)$, this implies

$\mu^*([-N,N] \cap S) = 0$. Thus, $[-N,N] \cap S$ is countable for every N, which implies S itself is countable. Similarly, if $[-1,1] \cap S$ is uncountable, then $\mu^*([-N,N] \setminus S) = 0$ for every N, which implies that $\mathbb{R} \setminus S$ is countable.

Thus, any σ-finite subset of $\mathbb{R}$ is μ^*-null, since any μ^*-measurable set with finite measure is, while there are μ^*-measurable sets S with $\mu^*(S) = \infty$, so not every μ^*-measurable set is σ-finite.

Finally, we see that μ^* cannot be induced by a measure μ on a ring $\mathcal{R}$. If it were, then by the reasoning in the solution to problem 5 there would be a μ^*-measurable set $G \supseteq [0,1]$ with $\mu^*(G) = 1$, which by the above is impossible.