Math 202B Solutions

Assignment 1

D. Sarason

1. Let $\mathcal{R}$ be a ring on a set X, let $\mathcal{R}'$ be the family of complements of the sets in $\mathcal{R}$, and let $\mathcal{A} = \mathcal{R} \cup \mathcal{R}'$. Prove that $\mathcal{A}$ is an algebra, and is a σ-algebra if $\mathcal{R}$ is a σ-ring.

   Proof: First, if $A, B \in \mathcal{R}$, and we set $A' = X \setminus A \in \mathcal{R}'$ and $B' = X \setminus B \in \mathcal{R}'$, then $A \cup B \in \mathcal{R}$; $A' \cup B' = X \setminus (A \cap B) \in \mathcal{R}'$; and $A \cup B' = X \setminus (B \setminus A) \in \mathcal{R}'$. Therefore, $\mathcal{A}$ is closed under finite unions. Since $\mathcal{A}$ is obviously closed under absolute complements, if $A, B \in \mathcal{A}$, then $A \setminus B = X \setminus (B \cup (X \setminus A)) \in \mathcal{A}$, showing that $\mathcal{A}$ is also closed under relative complements. Clearly, $X = X \setminus \emptyset \in \mathcal{R}'$, which completes the proof that $\mathcal{A}$ is an algebra.

   Now if $\mathcal{R}$ is a σ-ring, then $\mathcal{R}'$ is closed under countable unions, since $\mathcal{R}$ is closed under countable intersections. Namely, if $A_n \in \mathcal{R}$ for $n = 1, 2, \ldots$, then $\bigcup_{n=1}^{\infty}(X \setminus A_n) = X \setminus \bigcap_{n=1}^{\infty} A_n \in \mathcal{R}'$. Thus, if we have $A_n \in \mathcal{A}$ for $n = 1, 2, \ldots$, then $\bigcup_{A_n \in \mathcal{R}} A_n \in \mathcal{R}$, and $\bigcup_{A_n \in \mathcal{R}'} A_n \in \mathcal{A}$ (the latter union could be empty if all $A_n \in \mathcal{R}$). Therefore, $\bigcup_{n=1}^{\infty} A_n = \left(\bigcup_{A_n \in \mathcal{R}} A_n\right) \cup \left(\bigcup_{A_n \in \mathcal{R}'} A_n\right) \in \mathcal{A}$.

2. Let $\mathcal{L}$ be a lattice of sets, that is, a family of sets that contains $\emptyset$ and is closed under finite unions and finite intersections. Prove that the family of relative complements of the sets in $\mathcal{L}$ is a semiring.

   Proof: First, $\emptyset = \emptyset \setminus \emptyset$ is in this family. Now, for $A, B, C, D \in \mathcal{L}$, we have $(A \setminus B) \cap (C \setminus D) = (A \cap C) \setminus (B \cup D)$ is a relative complement of sets in $\mathcal{L}$. Similarly, $(A \setminus B) \setminus (C \setminus D)$ can be expressed as the disjoint union of $A \setminus (B \cup C)$ and $(A \cap C \cap D) \setminus B$.

3. Let X be a complete metric space, and let $\mathcal{A}$ be the family of subsets of X that are either meager or residual. For A in $\mathcal{A}$ define

$$\mu(A) = \begin{cases} 0 & \text{if } A \text{ is meager} \\ 1 & \text{if } A \text{ is residual.} \end{cases}$$

   Prove that $\mathcal{A}$ is a σ-algebra and that μ is a measure.

   Proof: Since the family of meager subsets of X is a σ-ring (in fact, a hereditary σ-ring), problem 1 implies that $\mathcal{A}$ is a σ-algebra. Also, by the Baire category theorem, no subset of X is both meager and residual, so μ is well-defined. Obviously, $\mu(\emptyset) = 0$. Now suppose $E_n \in \mathcal{A}$ are disjoint. Then if each E_n is meager, then $\bigcup_{n=1}^{\infty} E_n$ is meager, so $\mu(\bigcup_{n=1}^{\infty} E_n) = 0 = \sum_{n=1}^{\infty} \mu(E_n)$. Otherwise, some E_n is residual, which implies all other E_m are meager, since no two residual subsets of X are disjoint. Also, $\bigcup_{n=1}^{\infty} E_n$ is residual. Thus, $\mu(\bigcup_{n=1}^{\infty} E_n) = 1 = \sum_{n=1}^{\infty} \mu(E_n)$.

4. Let $X = \{0,1\}^{\mathbb{N}}$, the set of all sequences of 0's and 1's (aka the coin-tossing space), regarded as a topological space with the product topology (each coordinate space $\{0,1\}$ having the discrete topology). For n in $\mathbb{N}$ let P_n denote the n-th coordinate projection on X, the function that maps a sequence in X to its n-th term. Recall that the subbasic open sets in X are the sets $P_n^{-1}(\epsilon)$ $(n \in \mathbb{N}, \epsilon \in \{0,1\})$, and the basic open sets are the finite intersections of subbasic open sets.

   (a) Prove the Borel σ-algebra on X is the σ-algebra generated by the basic open sets.

   Proof: Since there are countably many subbasic open sets, there are only countably many possible finite intersections of them, so the given basis of X is countable. Now any open set in X is a union of basic open sets, and this union is automatically countable, which shows that every open set is in the σ-algebra generated by the basic open sets. Thus, the Borel σ-algebra on X is contained in the σ-algebra generated by the basic open sets, and the other inclusion is obvious.

   (b) Prove the basic open sets, together with $\emptyset$, form a semiring.

   Proof: For S a finite subset of $\mathbb{N}$ and $f : S \to \{0,1\}$, let $U(S,f)$ be the set of functions $x : \mathbb{N} \to \{0,1\}$ such that the restriction $x|S = f$. (This is another way of stating the definition given on the problem sheet.) Thus, the sets $U(S,f)$ enumerate the basic open sets. We now have $U(S,f) \cap U(T,g) = \emptyset$ if $f|(S \cap T) \neq g|(S \cap T)$. Otherwise, $U(S,f) \cap U(T,g) = U(S \cup T, f \cup g)$.

   Also, $U(S,f) \setminus U(T,g)$ is the disjoint union of sets $U(S \cup T, h)$ over functions $h : S \cup T \to \{0,1\}$ satisfying $h|S = f$ but $h|T \neq g$. Since $S \cup T$ is a finite set, there are finitely many such functions h.