Math 202B **Spring 2004**
Sarason

HOMEWORK ASSIGNMENT 8

Due in class on Friday, March 19.

29. Let μ be a signed measure on a σ-ring $\mathcal{R}$. Suppose μ_1 and μ_2 are nonnegative measures on $\mathcal{R}$, at least one of which is finite, such that $\mu = \mu_1 - \mu_2$. Prove that $\mu_+ \leq \mu_1$ and $\mu_- \leq \mu_2$ (i.e., $\mu_+(E) \leq \mu_1(E)$ and $\mu_-(E) \leq \mu_2(E)$ for every set E in $\mathcal{R}$).

30. Let $\mathcal{A}$ be a σ-algebra on a set X. Let $M(\mathcal{A})$ be the space of finite signed measure on $\mathcal{A}$, made into a real vector space under the operations

$$\begin{aligned}(c\mu)(E) &= c\mu(E) \ (\mu \in M(\mathcal{A}), c \in \mathbb{R}, E \in \mathcal{A}) \\ (\mu_1 + \mu_2)(E) &= \mu_1(E) + \mu_2(E) \ (\mu_1, \mu_2 \in M(\mathcal{A}), E \in \mathcal{A}).\end{aligned}$$

For μ in $M(\mathcal{A})$ let $\|\mu\| = |\mu|(X)$. Prove that $\|\cdot\|$ is a norm on $M(\mathcal{A})$, and that $M(\mathcal{A})$ is complete under this norm.

31. Let $M(\mathbb{R}^N)$ be the space of finite signed measures on $\mathcal{B}(\mathbb{R}^N)$, the Borel σ-algebra on $\mathbb{R}^N$, made into a normed space as in Problem 30. For μ and ν in $M(\mathbb{R}^N)$, define $\mu * \nu : \mathcal{B}(\mathbb{R}^N) \to \mathbb{R}$ by

$$\mu * \nu(E) = \int_{\mathbb{R}^N} \left(\int_{\mathbb{R}^N} \chi_E(x+y) d\nu(y) \right) d\mu(x),$$

in other words, $\mu * \nu(E) = (\mu \times \nu)(A_E)$, where $A_E = \{(x, y) \in \mathbb{R}^{2N} : x + y \in E\}$.

(a) Prove that $\mu * \nu$ is in $M(\mathbb{R}^N)$.

(b) Prove that if f is a bounded Borel-measurable function on $\mathbb{R}^N$, then

$$\int_{\mathbb{R}^N} f \ d(\mu * \nu) = \int_{\mathbb{R}^N} \left(\int_{\mathbb{R}^N} f(x+y) d\nu(y) \right) d\mu(x).$$

(c) Prove that $(\mu * \nu) * \xi = \mu * (\nu * \xi)$.

(d) Prove that $\|\mu * \nu\| \leq \|\mu\| \|\nu\|$.

(e) For f a Lebesgue-integrable function on $\mathbb{R}^N$, define the measure μ_f in $M(\mathbb{R}^N)$ by $\mu_f(E) = \int_E f \ d\lambda_N$. Prove that $\mu_f * \mu_g = \mu_{f*g}$ for any two Lebesgue-integrable functions f and g. (Thus, convolution of measures generalizes convolution in $L^1(\lambda_N)$.)