Math 202B **Spring 2004**
Sarason

HOMEWORK ASSIGNMENT 10

Due in class on Friday, April 9.

36. Let B_1 and B_2 be Banach spaces, with the norm in each denoted by $\|\cdot\|$. Let p be a number in $[1,\infty]$.

(a) Prove one gets a norm on $B_1 \oplus B_2$, the algebraic direct sum of B_1 and B_2, if one defines

$$\|x_1 \oplus x_2\| = \begin{cases} (\|x_1\|^p + \|x_2\|^p)^{1/p}, & 1 \le p < \infty \\ \max\{\|x_1\|, \|x_2\|), & p = \infty. \end{cases}$$

(b) Let $B_1 \oplus_p B_2$ denote $B_1 \oplus B_2$ equipped with the preceding norm. Prove $B_1 \oplus_p B_2$ is complete.

(c) Prove that the dual of $B_1 \oplus_p B_2$ equals $B_1^* \oplus_{p'} B_2^*$.

37. Prove that all norms on a finite-dimensional vector space B are equivalent: if $\|\cdot\|$ and $\|\cdot\|'$ are norms on B, then there are positive constants c_1 and c_2 such that

$$c_1\|x\| \le \|x\|' \le c_2\|x\|$$

for all x in B.

38. Prove that a finite dimensional subspace of a Banach space is closed.

39. (a) Let $x_1, \dots, x_n$ be linearly independent vectors in a Banach space B. Prove that there are functionals $\varphi_1, \dots, \varphi_n$ in B^* such that $\varphi_j(x_j) = 1$ for all j and $\varphi_j(x_k) = 0$ for $j \ne k$.

(b) Let A be a finite-dimensional subspace of a Banach space B. Prove that there is a closed subspace A' of B such that $A \cap A' = \{0\}$ and $A + A' = B$.

40. Consider the Banach space ℓ^∞ (real scalars). Let $T : \ell^\infty \to \ell^\infty$ be the shift operator on ℓ^∞, the map that sends $x = (x_1, x_2, \dots)$ in ℓ^∞ to $Tx = (0, x_1, x_2, \dots)$. Let $Y = \{x - Tx : x \in \ell^\infty\}$ and let e be the sequence $(1, 1, \dots)$.

(a) Prove $\operatorname{dist}(e, Y) = 1$.

(b) Prove there is a φ in $(\ell^\infty)^*$ such that $\varphi(e) = 1$, $\|\varphi\| = 1$, and $\varphi = 0$ on Y.

(c) Prove φ is translation invariant: $\varphi(Tx) = \varphi(x)$ for all x.

(d) Prove that $\liminf_{n\to\infty} x_n \le \varphi(x) \le \limsup_{n\to\infty} x_n$ for all x. (In particular, $\varphi(x) = \lim_{n\to\infty} x_n$ if x converges.) (Banach. Such a functional φ is called a Banach limit.)