Math 104, Section 6 **Fall 2003**
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REVIEW PROBLEMS

1. Let C be a countable set and F a nonempty finite set.

   (a) Prove that the set of functions of F into C is countable.

   (b) Prove that, if card $F > 1$, the set of functions of C into F is uncountable.

2. (a) Prove that the set of polynomials with integer coefficients is countable.

   (b) Use (a) to prove that the set of algebraic numbers is countable.

3. Prove that the set of increasing sequences of natural numbers has the same cardinality as the set of all sequences of natural numbers.

4. Prove that the sets $\mathbb{R}$, $(0,1)$, $[0,1]$, and $\mathbb{R}\backslash\mathbb{Q}$ all have the same cardinality.

5. Let $(a_n)_1^\infty$ be a sequence in $\mathbb{R}$ such that $\lim_{n\to\infty} a_n = 0$. Prove there is a sequence $(b_n)_1^\infty$ in $\mathbb{R}$ such that $\lim_{n\to\infty} b_n = \infty$ and $\lim_{n\to\infty} a_n b_n = 0$.

6. Let $(s_n)_1^\infty$ be a sequence in $\mathbb{R}$, and let the sequence $(t_n)_1^\infty$ be defined by $t_n = \frac{1}{n}(s_1+s_2+\cdots+s_n)$.

   (a) Find an example where $(s_n)_1^\infty$ diverges but $(t_n)_1^\infty$ converges.

   (b) Prove that if $(s_n)_1^\infty$ is monotone and $(t_n)_1^\infty$ converges, then $(s_n)_1^\infty$ converges.

7. Let A be a subset of $\mathbb{R}$, let A' denote its set of limit points, and let A'' denote the set of limit points of A'. Prove that $A'' \subset A'$, and $(A \cup A')' = A'$.

8. (a) Construct a subset A of $\mathbb{R}$ such that $A' \neq \emptyset$ but $A'' = \emptyset$.

   (b) Construct a subset A of $\mathbb{R}$ such that $A'' \neq \emptyset$ but $A''' = \emptyset$.

9. Let $(s_n)_1^\infty$ and $(t_n)_1^\infty$ be bounded sequences in $\mathbb{R}$, let A be the set of cluster points of $(s_n)_1^\infty$, let B be the set of cluster points of $(t_n)_1^\infty$, and let C be the set of cluster points of $(s_n+t_n)_1^\infty$. Prove that
$$C \subset A + B := \{a + b : a \in A,\ b \in B\}.$$
   Give an example in which the inclusion is strict.

10. Let $(s_n)_1^\infty$ be a bounded sequence in $\mathbb{R}$ and $(t_n)_1^\infty$ a convergent sequence. Prove that
$$\limsup_{n\to\infty}(s_n + t_n) = \limsup_{n\to\infty} s_n + \lim_{n\to\infty} t_n.$$

11. Let a and b be natural numbers. Define the sequence $(r_n)_1^\infty$ recursively by
$$r_1 = 1,\ r_{n+1} = a + \frac{b}{r_n}\ (n = 1, 2, \ldots).$$
    Prove that $(r_n)_1^\infty$ converges, and find the limit.

12. Let $(b_n)_1^\infty$ be a bounded increasing sequence in $\mathbb{R}$. Suppose the sequence $(s_n)_1^\infty$ satisfies $|s_{n+1} - s_n| \le b_{n+1} - b_n$. Prove $(s_n)_1^\infty$ converges.