Math 104, Section 1 **Fall 2004**
Sarason

Review Problems

1. Prove that every polynomial on $\mathbb{R}$ of odd degree has a real root.

2. Let $L_1, L_2, \ldots$ be lines in $\mathbb{R}^2$. Prove $\cup_{n=1}^{\infty} L_n \neq \mathbb{R}^2$.

3. Let E be an uncountable subset of $\mathbb{R}$. A point c of $\mathbb{R}$ is called a condensation point of E if $(c-\varepsilon, c+\varepsilon) \cap E$ is uncountable for every $\varepsilon > 0$. Let C be the set of condensation points of E. Prove $E \backslash C$ is countable.

4. Let the function $\varphi : \mathbb{R}^2 \to \mathbb{R}$ be defined by $\varphi(x, y) = x + y$.
   (a) Prove φ maps open sets onto open sets.
   (b) Find a closed subset of $\mathbb{R}^2$ whose image under φ is not closed.
   (c) Prove φ maps bounded closed subsets of $\mathbb{R}^2$ onto closed sets.

5. Let K be a compact subset of $\mathbb{R}^N$ and F a closed subset of $\mathbb{R}^N$. Prove the set $K + F = \{x + y : x \in K,\ y \in F\}$ is closed.

6. Let M be a metric space, K a compact subset of M, and $(f_n)_1^{\infty}$ a sequence of nonnegative, continuous, real-valued functions on K satisfying the conditions (i) $f_{n+1}(x) \leq f_n(x)$ for all n and all x in K, and (ii) $\lim_{n\to\infty} f_n(x) = 0$ for all x in K. Prove $f_n \to 0$ uniformly on K. (Dini's lemma)

7. Prove that all norms on $\mathbb{R}^N$ are equivalent to the Euclidean norm: If $\|\cdot\|$ is a norm on $\mathbb{R}^N$ then there are positive numbers a and b such that $a\|x\|_2 \leq \|x\| \leq b\|x\|_2$ for all x.

8. Prove or find a counterexample: If $f : (\alpha, \beta) \to \mathbb{R}$ is differentiable and c is a point of (α, β), then there are points a in (α, c) and b in (c, β) such that
$$f'(c) = \frac{f(b) - f(a)}{b - a}.$$

9. Let the function $F : [a, b] \times [\alpha, \beta] \to \mathbb{R}$ be continuous, and let the function $f : [a, b] \to \mathbb{R}$ be defined by
$$f(x) = \int_{\alpha}^{\beta} F(x, \xi)\, d\xi.$$
Prove f is continuous.

10. Let $f : [a, b] \to \mathbb{R}$ be bounded and $g : [a, b] \to \mathbb{R}$ Riemann integrable. Assume $|f(x) - g(x)| \leq \varepsilon$ for all x. Prove $U(f) - L(f) \leq 2\varepsilon(b - a)$.

11. Prove that if f and g are Riemann integrable on $[a, b]$ then fg is Riemann integrable on $[a, b]$. (Suggestion: Treat first the case $f = g$.)

12. Let f be a Riemann-integrable function on $[a, b]$. Prove that, for every $\varepsilon > 0$, there are continuous functions g and h such that $g \leq f \leq h$ and $\int_a^b (h - g) < \varepsilon$. (Suggestion: Treat first the case where f is a step function.)

13. Use the Weierstrass approximation theorem to prove that $C[a,b]$ is separable.

14. Prove there is a sequence $(p_n)_1^\infty$ of polynomials such that

$$\lim_{n\to\infty} p_n(x) = \begin{cases} 1, & x > 0 \\ 0, & x = 0 \\ -1, & x < 0. \end{cases}$$

15. (a) Let f be a continuous real-valued function on $[0,1]$ such that $\int_0^1 x^n f(x) = 0$ for $n = 0, 1, 2, \ldots$. Prove $f = 0$.

    (b) Suppose f is as in (a) but it is only assumed that $\int_0^1 x^n f(x)dx = 0$ for $n = 0, 2, 4, \ldots$. Can you still conclude $f = 0$?

16. Let K be a compact metric space and $(f_n)_1^\infty$ a sequence of functions in $C(K)$ with the properties (i) the set $\{f_n : n \in \mathbb{N}\}$ is equicontinuous, and (ii) there is a dense subset S of K such that the sequence $(f_n(x))_1^\infty$ converges for each x in S. Prove the sequence $(f_n)_1^\infty$ converges uniformly on K.