Math 104, Section 1 **Fall 2004**
Sarason

Review Problems

1. Let f be a continuous increasing function from the open interval (a, b) into $\mathbb{R}$.
   (a) Prove the range of f is an open interval.
   (b) Prove the inverse function f^{-1} (whose domain is the range of f) is continuous

2. Let (M, d) be a metric space. Let the function $\lambda = [0, \infty) \to \mathbb{R}$ be continuous, increasing, vanishing at 0, and satisfy $\lambda(s + t) \leq \lambda(s) + \lambda(t)$ for all s and t. Define the function $d_\lambda : M \times M \to \mathbb{R}$ by $d_\lambda(x, y) = \lambda(d(x, y))$.
   (a) Prove d_λ is a metric on M.
   (b) Prove the metric spaces (M, d) and (M, d_λ) have the same open sets.

3. Let the function $f : (0, 1] \to \mathbb{R}$ be uniformly continuous. Prove $\lim_{x \to 0} f(x)$ exists.

4. Let M and N be metric spaces and $f : M \to N$ and $g : M \to N$ continuous functions.
   (a) Is the subset $\{x \in M : f(x) = g(x)\}$ of M necessarily closed? Is it necessarily connected?
   (b) Is the subset $\{(f(x), g(x)) : x \in M\}$ of $N \times N$ necessarily closed? Is it necessarily connected?

5. Let A and B be subsets of a metric space.
   (a) Prove $\operatorname{int}(A \cap B) = \operatorname{int}(A) \cap \operatorname{int}(B)$.
   (b) Find an example for which $\operatorname{int}(A \cup B) \neq \operatorname{int}(A) \cup \operatorname{int}(B)$.

6. Let G be a nonempty open subset and H a nonempty subset of $\mathbb{R}^N$. Prove the set $G + H = \{x + y : x \in G,\ y \in H\}$ is open.

7. Let the continuous function $f : \mathbb{R}^N \to \mathbb{R}^M$ have the property that $f^{-1}(K)$ is a compact subset of $\mathbb{R}^N$ whenever K is a compact subset of $\mathbb{R}^M$. Prove $f(C)$ is a closed subset of $\mathbb{R}^M$ whenever C is a closed subset of $\mathbb{R}^N$.

8. (a) Suppose the graph of the function $f : \mathbb{R} \to \mathbb{R}$ is closed and connected. Prove f is continuous.
   (b) Find a discontinuous function from $\mathbb{R}$ to $\mathbb{R}$ whose graph is connected.

9. Let C be a nonempty subset of $\mathbb{R}^N$ with the property that every continuous function of C into $\mathbb{R}$ attains a maximum value on C. Prove C is compact.

10. Let $(a_n)_1^\infty$ be a convergent sequence in a metric space, with limit a_0. Prove the set $\{a_0, a_1, a_2, \dots\}$ is compact.

11. Let f be a function from the metric space M into the metric space N. Prove f is continuous if and only if $f(\overline{A}) \subset \overline{f(A)}$ for every subset A of M.

12. Let S be the union of the set of lines in $\mathbb{R}^2$ that have rational slope and pass through either $(1,0)$ or $(0,1)$. Prove S is connected.

13. Let $(K_n)_1^\infty$ be a nested sequence of nonempty compact subsets of a metric space, and let $K = \bigcap_{n=1}^{\infty} K_n$. Prove $\operatorname{diam} K = \lim_{n\to\infty} \operatorname{diam} K_n$.

14. Let M and N be metric spaces and $(f_n)_1^\infty$ a uniformly convergent sequence in $C(M,N)$ with limit f. Let $(x_n)_1^\infty$ be a convergent sequence in M with limit x_0. Prove $f(x_0) = \lim_{n\to\infty} f_n(x_n)$.

15. Let K be the set of functions f in $C[0,1]$ that satisfy (a) $\|f\|_\infty \leq 1$, and (b) $|f(x) - f(y)| \leq |x-y|$ for all x and y in $[0,1]$. Prove K is compact. (Suggestion: Given a sequence of functions in K, prove it has a subsequence that converges pointwise on $\mathbb{Q} \cap [0,1]$. Then prove the subsequence actually converges uniformly on $[0,1]$.