Math 104, Section 1 **Fall 2004**
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Review Problems

1. Prove that the set of nondecreasing functions of $\mathbb{N}$ onto $\{0, 1, 2\}$ is denumerable.

2. Let $\mathcal{A}$ be a family of nonempty subsets of $\mathbb{N}$ such that the intersection of any two distinct sets in $\mathcal{A}$ is either empty or a singleton. Prove $\mathcal{A}$ is countable.

3. Let $\mathcal{S}$ be the set of functions $f : \mathbb{Q} \to \mathbb{Q}$ that satisfy $f(a + b) = f(a) + f(b)$ for all a and b in $\mathbb{Q}$. Prove $\mathcal{S}$ is denumerable.

4. Prove that the set of functions of $\mathbb{N}$ into $\mathbb{N}$ has the same cardinality as the set of increasing functions of $\mathbb{N}$ into $\mathbb{N}$.

5. Let $(a_n)_1^\infty$ be a sequence in $\mathbb{R}$ with limit 0. Prove there is a sequence $(b_n)_1^\infty$ in $\mathbb{R}$ such that $\lim_{n\to\infty} |b_n| = \infty$ and $\lim_{n\to\infty} a_n b_n = 0$.

6. Let $(b_n)_1^\infty$ be a bounded increasing sequence in $\mathbb{R}$. Let the sequence $(a_n)_1^\infty$ satisfy $|a_{n+1} - a_n| \leq b_{n+1} - b_n$ for all n. Prove $(a_n)_1^\infty$ converges.

7. Let $(a_n)_1^\infty$ be a monotone sequence in $\mathbb{R}$ such that the sequence $(b_n)_1^\infty$ defined by $b_n = \frac{1}{n}(a_1 + a_2 + \cdots + a_n)$ converges. Prove $(a_n)_1^\infty$ converges.

8. Let $(a_n)_1^\infty$ be a sequence in $\mathbb{R}$ such that $\lim_{n\to\infty}(a_{n+k} - a_n) = 0$ for each k in $\mathbb{N}$. Can you conclude that $(a_n)_1^\infty$ is a Cauchy sequence?

9. Let $(a_n)_1^\infty$ be a bounded divergent sequence in $\mathbb{R}$. Prove there are two convergent subsequences of $(a_n)_1^\infty$ with different limits.

10. Let S be an uncountable subset of $\mathbb{R}$. Prove there is a real number a such that the set $S \cap (a - \varepsilon, a + \varepsilon)$ is uncountable for every $\varepsilon > 0$.