Math 104, Section 1 **Fall 2004**
Sarason

HOMEWORK ASSIGNMENT 9

Due in class on Wednesday, November 17.

37. Let the function $f : (0,1) \to \mathbb{R}$ be continuously differentiable. Let the function $g : (0,1) \times (0,1) \to \mathbb{R}$ be defined by
$$g(x,y) = \begin{cases} \frac{f(x)-f(y)}{x-y}, & x \neq y \\ f'(x), & x = y. \end{cases}$$
Prove g is continuous.

38. Let I be an open interval and $f : I \to \mathbb{R}$ a differentiable function such that f' is nondecreasing.
    (a) Prove f' is continuous.
    (b) Prove f is convex, i.e., for a and b in I with $a < b$ and $0 < t < 1$,
$$f((1-t)a + tb) \leq (1-t)f(a) + tf(b).$$

39. Let I be an open interval and $f : I \to \mathbb{R}$ a twice differentiable function, with f'' continuous. Prove that, for x_0 in I,
$$f''(x_0) = \lim_{\delta \to 0} \frac{f(x_0+\delta) + f(x_0-\delta) - 2f(x_0)}{\delta^2}.$$

40. Let the sequence $(f_n)_1^\infty$ of Riemann integrable functions on the interval $[a,b]$ converge uniformly to the function f. Prove f is Riemann integrable, and
$$\int_a^b f = \lim_{n\to\infty} \int_a^b f_n.$$
(Remark: This is proved in Pugh, pp. 207–208, but the proof there uses a deep result of Lebesgue characterizing Riemann integrability which we have not taken up. The result can be proved without resort to Lebesgue's theorem. You are asked to find such a proof.)