Math 104, Section 1 **Fall 2004**
Sarason

HOMEWORK ASSIGNMENT 6

Due in class on Wednesday, October 20.

22. Prove that if S_1 and S_2 are subsets of a metric space then $(S_1 \cup S_2)' = S_1' \cup S_2'$.

23. Let F be a closed subset of the separable metric space M. Prove there is a countable set whose closure is F.

24. The diameter of a nonempty subset S of a metric space is defined by

$$\text{diam } S = \sup\{d(x,y) : x, y \in S\}.$$

Prove that if S is compact then there are points x_0 and y_0 in S such that diam $S = d(x_0, y_0)$.

25. For A and B nonempty subsets of a metric space M, the distance between them is defined by

$$\text{dist}(A,B) = \inf\{d(x,y) : x \in A,\ y \in B\}.$$

(a) Prove that if $M = \mathbb{R}^N$, A is compact, and B is closed, then there are points a in A and b in B such that $\text{dist}(A,B) = d(a,b)$.

(b) Find a pair of closed subsets A and B of $\mathbb{R}$ such that $\text{dist}(A,B) < d(x,y)$ for every x in A and y in B.

26. Let M and N be metric spaces. Make $M \times N$ into a metric space by defining

$$d_{M\times N}((x_1,y_1),(x_2,y_2)) = d_M(x_1,x_2) + d_N(y_1,y_2).$$

(a) Prove that if the function $f : M \to N$ is continuous then the graph of f is a closed subset of $M \times N$.

(b) Find an example of a discontinuous function with a closed graph, but show that if N is compact then a function from M to N whose graph is closed is necessarily continuous.

(For simplicity, denote $d_M, d_N, d_{M\times N}$ all by d in your write-up.)