M. Rieffel Math 202A Final Exam
December 17, 2014

SHOW YOUR WORK COMPLETELY AND NEATLY.
DON'T WRITE MORE THAN YOU NEED TO.
YOU CAN USE THE BACKS OF PAGES, IF YOU LABEL YOUR WORK. Total points = 70.

Name ....

SID# ....

Scores:

problem 1 ....

problem 2 ....

problem 3 ....

problem 4 ....

problem 5 ....

problem 6 ....

problem 7 ....

problem 8 ....

1. a) (2 points) State Urysohn's Lemma.

b) (6 points) Use Urysohn's Lemma to prove that if X is a locally compact Hausdorff space and if C is a compact subset of X then there is a continuous function, f, of compact support such that $f(x) = 1$ for all $x \in C$. (You can use other facts that we proved in class about locally compact spaces.)

2. (6 points) Prove that if X and Y are locally compact Hausdorff spaces then $X \times Y$, equipped with the product topology, is locally compact.

3. (6 points) Let X and Y be compact Hausdorff spaces. Prove that the linear span (i.e. the set of finite linear combinations) of functions on $X \times Y$ of the form $F(x, y) = f(x)g(y)$ for $f \in C(X)$ and $g \in C(Y)$ is dense in $C(X \times Y)$ for the uniform norm.

4. Let X be a topological space, and let (M, d) be a metric space.

a) (2 points) Define what it means for a family $\mathcal{F}$ of functions from X to M to be *equicontinuous*.

b) (8 points) Give a proof of that part of the Arzela-Ascoli theorem that says that if X is compact and if $\mathcal{F}$ is an equicontinuous family of functions from X to M such that for each $x \in X$ the set $S_x = \{f(x) : f \in \mathcal{F}\}$ is a totally bounded subset of M, then $\mathcal{F}$ is totally bounded for the uniform metric d_∞ defined by $d_\infty(f, g) = \sup\{d(f(x), g(x)) : x \in X\}$.

5. Let $\{x_\lambda\}_{\lambda\in\Lambda}$ be a net in a topological space X.

a) (2 points) Define what is meant by a *cluster point* of this net.

b) (6 points) For any $\mu \in \Lambda$ let $A_\mu = \{x_\lambda : \lambda \geq \mu\}$, and let $\bar{A}_\mu$ be the closure of A_μ. Prove that $\bigcap_{\mu\in\Lambda} \bar{A}_\mu$ is the set of cluster points of the net, and thus the set of cluster points is closed.

6. Let $\mathcal{H}$ be a hereditary σ-ring of subsets of a set X.

a) (2 points) Define what is meant by an *outer measure* μ^* on $\mathcal{H}$.

b) (2 points) For an outer measure μ^* on $\mathcal{H}$ define what is meant by a μ^**-measurable* set in $\mathcal{H}$.

c) (7 points) Prove that the union of two μ^*-measurable sets in $\mathcal{H}$ is μ^*-measurable.

7. (7 points) Let $\mathcal{P}$ be the usual semi-ring of half-open intervals $[a, b)$ in $\mathbb{R}$, and define the function μ on $\mathcal{P}$ by $\mu([a, b)) = b - a$. Prove that μ is countably additive.

8. Let X be a compact Hausdorff space, and let $C(X)$ be the usual algebra of continuous $\mathbb{R}$-valued functions on X, equipped with the uniform norm $\|\cdot\|_\infty$.

a) (2 points) For each $f \in C(X)$ let $T_f = [-\|f\|_\infty, \|f\|_\infty]$, a closed interval in $\mathbb{R}$. Notice that the range of f is contained in T_f. Let $Y = \Pi\{T_f : f \in C(X)\}$, equipped with the product topology. State what the usual sub-base for the product topology is for the case of this particular example.

b) (4 points) A positive Radon measure ϕ on X (i.e. a positive linear functional on $C(X)$) is said to be a Radon *probability* measure if $\phi(1) = 1$ (where the first 1 is the constant function with value 1 on X). Let $\mathcal{P}(X)$ denote the set of Radon probability measures on X. Prove that if $\phi \in \mathcal{P}(X)$ then for any $f \in C(X)$ we have $\phi(f) \in T_f$ (for T_f as in part a)).

c) (2 points) Explain how the result obtained in part b) means that any $\phi \in \mathcal{P}(X)$ can be viewed as an element of the Y of part a), so that $\mathcal{P}(X)$ can be viewed as a subset of Y.

d) (6 points) Prove that $\mathcal{P}(X)$, when viewed as a subset of Y, is a closed subset of Y. (Tychonoff's theorem tells us that Y is compact. Since $P(X)$ is a closed subset of Y, it follows that $\mathcal{P}(X)$ itself is compact. This is important for a variety of applications of probability measures.)