M. Rieffel Math 202A, Homework 9 due October 26, 2018

1. Let $T = \{z \in \mathbb{C} : |z| = 1\}$, the unit circle. It is a topological group under multiplication. Define the group homomorphism $e : \mathbb{R} \to T$ by $e(r) = \exp(2\pi i r)$. Let μ be Lebesgue measure on $\mathbb{R}$ with M the collection of Lebesgue measurable sets. Set $M(T) = \{E \subseteq T : e^{-1}(E) \in M\}$. Show that $M(T)$ is a σ-ring. Define ν on $M(T)$ by $\nu(E) = \mu(e^{-1}(E) \cap [0, 1))$. Show that ν is a measure, and that ν is "translation-invariant", i.e. "rotation-invariant", in the evident sense.

2. Let α be the non-decreasing function on $\mathbb{R}$ defined by $\alpha(t) = 0$ if $t \leq 0$ and $\alpha(t) = 1$ if $t > 0$. Let μ_α be the corresponding premeasure as discussed in lecture, with μ_α^* the corresponding outer measure. Determine which sets are in $M(\mu_\alpha^*)$, that is, measurable for this outer measure.

3. Let α be the function on $\mathbb{R}$ defined by $\alpha(\mathrm{r}) = \mathrm{r}$, which is the function that leads to Lebesgue measure on $\mathbb{R}$. Let μ_α be the corresponding premeasure, and Let μ_α^* be the corresponding outer measure on $\mathbb{R}$. Let Q be the set of rational numbers, viewed as a subset of $\mathbb{R}$. Calculate $\mu_\alpha^*(Q)$.