M. Rieffel Math 202A, Homework 8 due October 19, 2018

1. Let α be a non-decreasing function on $\mathbb{R}$.

a) As in my lectures, define μ_α on the semiring $\mathcal{P}$ of left-closed right-open finite intervals $[a, b)$ by $\mu_\alpha([a, b)) = \alpha(b) - \alpha(a)$. Show by example that if α is not left-continuous, then μ need not be countably additive on $\mathcal{P}$.

b) Assume now that $\alpha(\mathrm{r}) = \mathrm{r}$, which is the function that leads to Lebesgue measure. But now take $\mathcal{P}$ to be the family of intervals $[a, b)$ in $\mathcal{Q}$, the field of rational numbers (with $a, b \in \mathcal{Q}$). I.e. pretend you have never heard of the real numbers. Define μ_α as above. Show by example that μ_α is not countably additive on $\mathcal{P}$.

2. Let $\mathcal{T}$ be a family of subsets of a set X which is closed under taking finite unions and finite intersections (e.g. the family of compact subsets of a topological space, or the family of open subsets of a topological space.) Let $\mathcal{P}$ denote the family of differences, $E \setminus F$, of elements of $\mathcal{T}$. Prove that $\mathcal{P}$ is a semiring. (So this might be used to define measures on topological, and especially locally compact, spaces.)

3. Let $\mathcal{P}_1$ and $\mathcal{P}_2$ be semirings of subsets of two sets X_1 and X_2, and let $\mathcal{P}_1 \times \mathcal{P}_2$ denote the family of all products ("rectangles"), $E_1 \times E_2$, of elements $E_1 \in \mathcal{P}_1$ and $E_2 \in \mathcal{P}_2$. Prove that $\mathcal{P}_1 \times \mathcal{P}_2$ is a semiring of subsets of $X_1 \times X_2$. (So you have "product measures" in your future.)

4. Let $\mathcal{F}$ be a family of subsets of a set X. Let $R(\mathcal{F})$ be the ring generated by $\mathcal{F}$, and let $S(\mathcal{F})$ be the σ-ring generated by $\mathcal{F}$.

a) Prove that every element of $R(\mathcal{F})$ is contained in a finite union of elements of $\mathcal{F}$.

a) Prove that every element of $S(\mathcal{F})$ is contained in a countable union of elements of $\mathcal{F}$.