M. Rieffel Math 202A, Homework 7 due October 12, 2018

1. a) Equip $\mathbb{R}^n$ with its usual Euclidean metric, and let $Ball$ be the closed unit ball in $\mathbb{R}^n$. Prove that for any normal topological space X and any closed subset A of X and any continuous function f from A into $Ball$ there is an extension of f to a continuous function from all of X into $Ball$. (Use the Tietze extension theorem as a main tool.)

b) What do you think are the chances that the above fact also holds if "closed ball" is replaced by "sphere" (i.e. just the surface of the ball)? No proof needed, but give a specific example for the case of the sphere in $\mathbb{R}^3$ that you expect supports your guess. (Contemplate how you might prove your guess.)

c) Call the ball and the sphere in the above situation "targets". Show that the fact you prove in part a) is still true if the closed ball is replaced as target by a topological space that is homeomorphic to the closed ball? What about in part 2 replacing the sphere as target by a topological space that is homeomorphic to the sphere? Thus notice that whether a topological space can be the target for a Tietze-extension-type theorem depends only on its topology, and not on any metric considerations.

2. Let $V = C([0,1])$, equipped with the norm $\|\cdot\|_\infty$. Define a linear operator, T, from V to itself by

$$(T(f))(t) = \int_0^t f(s)\, ds.$$

Prove that the image under T of the unit ball in V is totally bounded in V. (Thus we say that T is a "compact operator".)

3. a) For a metric space (X, d) let $\mathcal{L}(X)$ denote the set of $\mathbb{R}$-valued Lipschitz functions defined on X. For $f \in \mathcal{L}(X)$ let $L(f)$ denote the Lipschitz constant of f, that is, the smallest constant c such that

$$|f(x_1) - f(x_2)| \leq c\, d(x_1, x_2)$$

for all $x_1, x_2 \in X$. Show that $\mathcal{L}(X)$ is a (perhaps not closed) linear subspace of $C(X)$, and that L is a seminorm on $\mathcal{L}(X)$.

b) Assume now that the metric space (X, d) is a compact. Define a norm, $\|\cdot\|_L$, on $\mathcal{L}(X)$ by

$$\|f\|_L = \|f\|_\infty + L(f).$$

Prove that $\mathcal{L}(X)$ is complete for this norm.

c) For (X, d) still compact, let S be the linear operator from $\mathcal{L}(X)$ to $C(X)$ that sends each $f \in \mathcal{L}(X)$ to f viewed as an element of $C(X)$. Prove that the image under S of the unit ball of $\mathcal{L}(X)$ (for the norm $\|\cdot\|_L$) is totally bounded in $C(X)$ (with its uniform norm). (Thus S is a compact operator.)

d) Let $\mathcal{A}$ be the algebra of infinitely differentiable functions of period 1 on $\mathbb{R}$, viewed as functions on the quotient space $T = [0,1]$ with $0 \sim 1$, and thus as the "smooth" functions on the circle, so $\mathcal{A}$ is a subalgebra of $C(T)$. Equip $C(T)$ with the supremum norm. Define a norm, $\|\cdot\|^1$, on $\mathcal{A}$ by

$$\|f\|^1 = \|f\|_\infty + \|f'\|_\infty$$

where f' denotes the derivative of f. (You can contemplate what the completion of $\mathcal{A}$ is for this norm.) Let S be the linear operator from $\mathcal{A}$ to $C(T)$, that sends each $f \in \mathcal{A}$ to f viewed as an element of $C(T)$. Prove that the image under S of the unit ball of $\mathcal{A}$ (for the norm $\|\cdot\|^1$) is totally bounded in $C(T)$ (with its uniform norm). (Thus S is a compact operator. There are versions of this involving higher derivatives, and functions of several variables, and norms from inner products, which in particular lead to "Sobolev spaces" (see Wikipedia), which are widely used in dealing with partial differential equations, and for which the corresponding total boundedness results are quite important.)