M. Rieffel   Math 202A, Homework 6   due October 5, 2018

1. Let I be the interval $[0, 1]$ in $\mathbb{R}$ with its usual topology. Let $B(I)$ denote the set of all functions from I to I. We can also consider I to be an index set, and if we set $X_i = I$ for all $i \in I$ then we can form the product space $X = \Pi_i X_i$, equipped with the product topology (so X is compact and Hausdorff). In an evident natural way we can identify $B(I)$ with X and so give $B(I)$ the corresponding topology.

a) Let $\{f_n\}$ be a sequence of functions in $B(I)$ that converges pointwise to a function $f \in B(I)$. Show that this sequence converges to f for the product topology. (For this reason, the product topology is sometimes referred to as the "topology of pointwise convergence". However, see the next parts of this problem.)

b) Let $F(I)$ denote the subset of $B(I)$ consisting of functions of finite support, i.e. functions that take the value 0 at all but a finite number of points of I. Prove that $F(I)$ is dense in $B(I)$, i.e. that its closure is $B(I)$.

c) Let $C(I)$ denote the subset of $B(I)$ consisting of continuous functions. Prove that if $f \in C(I)$ and f has a non-zero value at some point of I, then there is no sequence of elements of $F(I)$ that converges to f for the product topology.

d) Prove that $C(I)$ is dense in $B(I)$.

e) Prove that there is a countable subset of $C(I)$ that is dense in $C(I)$ for the relative topology, and so is dense in $B(I)$. Hint: consider polynomials with rational coefficients.

e) Let $D(I)$ denote the subset of $B(I)$ consisting of the "delta functions", i.e. the functions that take value 1 at one point and value 0 at all other points of I. Determine the relative topology on $D(I)$. Is there a countable subset of $D(I)$ that is dense in $D(I)$?

f) Determine the closure of $D(I)$ in $B(I)$.

2. Let A be an index set, and let $\{X_\alpha, \mathcal{T}_\alpha\}$ be a collection of topological spaces indexed by A. Let X denote the disjoint union of the X_α's. For each α there is a "canonical", i.e. "obvious natural" injective function i_α from X_α into X. The final topology on X for all of these functions is called the "direct sum" topology on X.

a) Describe the open sets for the direct sum topology.

b) Determine necessary and sufficient conditions in terms of the X_α's for X to be compact.

c) Determine necessary and sufficient conditions in terms of the X_α's for X to be Hausdorff.

3. Let (X, d_X) and (Y, d_Y) be metric spaces. Prove that if X is compact, then every continuous function from X to Y is uniformly continuous.