M. Rieffel Math 202A, Homework 5 due September 28, 2018

Connectedness is an important property that a topological space may have. But it is a quite simple property to deal with, so, given our time constraints, I plan not to lecture on it in class. Some of the following problems have proofs in the recommended texts, but I strongly encourage you to do the problems below without looking at any texts, as you will learn much more that way. This can also serve as a good review of some of the main topics that we have covered so far in the course.

Definition: A topological space is "connected" if it is not the union of two disjoint non-empty open subsets (which would then be "clopen" subsets).

A subset of a topological space is said to be connected if it is connected for the relative topology.

You have probably seen a proof of the fact that any interval of $\mathbb{R}$ is connected, and that this fails for intervals in the set of rational numbers with its metric topology. Review those proofs (or find you own proofs).

For many of the problems below you will probably find the concept of a *clopen* subset (i.e. a subset that is both open and closed) useful.

1) Prove that a continuous function from one topological space into another carries connected subsets onto connected subsets. (When applied to $\mathbb{R}$-valued functions, this gives the "Intermediate Value Theorem".)

2) Prove that the closure of a connected subset of a topological space is connected.

3) Prove that if $\mathcal{F}$ is a (possibly infinite) collection of connected subsets of a topological space, and if there is a point x_0 that is contained in every element of $\mathcal{F}$, then the union of all the elements of $\mathcal{F}$ is a connected subset.

4) Let X be a topological space. Declare that two points of X are equivalent if there is some connected subset of X that contains both of them. Prove that this is an equivalence relation. The equivalence classes are called the "connected components" of X. Prove that each connected component is a connected subset of X which is closed.

5) Determine the connected components of the set of rational numbers with its metric topology. (What can you conclude about whether connected components must be open?)

6) A topological space X is said to be "path-connected" if for any two points $x_0, x_1 \in X$ there is a continuous function p from a closed interval $[a, b]$ of $\mathbb{R}$ into X (a "path" in X) such that $p(a) = x_0$ and $p(b) = x_1$ (a path from x_0 to x_1). Prove that a path-connected topological space is connected.

7) Let X be a topological space. Declare that two points of X are equivalent if there is a path in X from one to the other. Prove that this is an equivalence relation. The equivalence

classes are called the "path-connected components" of X. Prove that each path-connected component is path-connected.

8) Let $A \subset \mathbb{R}^2$ be the union of the y-axis and the graph of the function $f(t) = \sin(1/t)$ for $0 < t \leq 1$. (Draw a picture of A.) Prove that A, with the relative topology, is connected but not path-connected. What are the path-connected components of A. What can you conclude about whether path-connected components must be closed?

9) Let X and Y be topological spaces, and let π be a continuous function from X onto Y such that the topology of Y is the quotient topology from X. Prove that if Y is connected, and if the pre-image in X of each point of Y for π is connected, then X is connected.

10) Let X and Y be connected topological spaces. Prove that $X \times Y$ with the product topology is connected.