M. Rieffel Math 202A, Homework 4 due September 21, 2018

1. Given two sets, X and Y, and a function, f, from X to Y, the graph of f, denoted $\mathrm{Gr}(f)$, is the subset $\{(x, f(x)) : x \in X\}$ of $X \times Y$. Suppose now that X and Y are topological spaces, and that Y is Hausdorff. Prove that if f is continuous then its graph is closed for the product topology on $X \times Y$. Give an example of an $\mathbb{R}$-valued function on $\mathbb{R}$ which is not continuous and yet whose graph is closed.

2. For any given positive integer p let $S_p = \{1, 2, \cdots, p\}$, the "set of symbols". For each $k \in \mathbb{Z}$ let $X_k = S_p$, and form the product space $Y_p = \prod_{k=-\infty}^{\infty} X_k$. It thus consists of all functions from $\mathbb{Z}$ to S_p. Equip S_p, and so each X_k with the discrete topology, and then equip Y_p with the corresponding product topology.

a) Let B denote the usual base for the product topology, for the case of the above Y_p. Show that each element of B is actually closed. (In this context the elements of the usual base are often called "cylinder sets". A subset of a topological space is said to be "clopen" if it is both open and closed. A topological space whose topology has a base consisting entirely of clopen sets is said to be "0-dimensional". Thus what you are being asked to show is that the above space Y_p is 0-dimensional.)

b) Define a function, T, from Y_p to itself by $(Tx)_k = x_{k-1}$ for each $x \in Y_p$. Show that T is a homeomorphism of Y_p with itself. (Usually T is called the "right shift" on Y_p. In some contexts T is called the "full Bernoulli shift". A movie recorded on a DVD disk consists of a very long string of 0's and 1's, corresponding to pits burned in the DVD disk. It is useful to consider these strings to be infinitely long, and so to be elements of X_2 (using 1's and 2's instead of 0's and 1's). This view is useful in considering different schemes for compression of data, etc. In probability theory Y_2 is useful in considering "coin-tossing" questions, often with a probability "measure" on S_p and a corresponding measure on Y_2.)

c) For each positive integer n we let T^n be the composition of T with itself n times. In the same way we define T^n for negative n by using T^{-1}. Show that the function $n \mapsto T^n$ is a group homomorphism of $\mathbb{Z}$ into the group of homeomorphisms from Y_p to itself. In this way we define an "action" of $\mathbb{Z}$ on Y_p. (Very often the integers are viewed as discrete time, the "ticks of the clock", and the system consisting of Y_p and the action of $\mathbb{Z}$ given by T is considered to be a "discrete" dynamical system.)

d) Given the above action of $\mathbb{Z}$ on Y_p, we can consider the orbits of points in Y_p. For any $x \in Y_p$ its orbit is $\{T^n(x) : n \in \mathbb{Z}\}$. We will soon see in class that Y_p is compact, and so orbits are likely to have limit points. Let x be the specific element of Y_2 defined by $x_1 = 2$, $x_3 = 2$, and $x_k = 1$ if $k \neq 1$ and $k \neq 3$. Determine the closure of the orbit of this x.

e) Find a nice characterization of the elements of Y_p whose orbits are finite. Then prove that the set of elements of Y_p whose orbits are finite is dense in Y_p.

f) Let W be the subset of Y_2 consisting of elements for which a 2 is always followed by a 1, that is, if $x_k = 2$ then $x_{k+1} = 1$. Prove that W is a closed subset of Y_2. Show that W is carried into itself by T and T^{-1}, and so by the action of $\mathbb{Z}$ defined in part c). (Then W with this action is an example of a "subshift of finite type". There are important classes of dynamical systems that are usefully modeled by various subshifts of finite type. This topic is called "symbolic dynamics". I strongly encourage you to glance at the Wikipedia entries for "symbolic dynamics" and "subshift of finite type".)

3. A function f from one topological space to another is said to be *open* if $f(O)$ is open for every open set O. The function f is said to be *closed* if $f(C)$ is closed for every closed set C.
Let $X = [0,1] \times \mathbb{R}$, with the product topology (which is the same as its relative topology as a subset of $\mathbb{R}^2$). Let π be the usual projection from X onto $[0,1]$. Let $\mathcal{S}$ be the collection of all subsets, A, of X such that $\pi(A) = [0,1]$, and let π_A denote the restriction of π to A. Give each $A \in \mathcal{S}$ its relative topology from X. Find elements of $\mathcal{S}$ such that:

a) π_A is both open and closed, but is not one-to-one.

b) π_A is not open but is closed.

c) π_A is open but not closed.

d) π_A is not open and not closed.

For each of your examples determine the quotient topology on $[0,1]$ coming from viewing $[0,1]$ as a quotient of A via π_A.