M. Rieffel Math 202A, Problem set 3
due September 14, 2018

1. Let X be the square $[-1,1]\times[-1,1]$, with its usual product topology. We "glue" together parts of the boundary of X by the following equivalence relation on X. Each point of the interior is its own equivalence class, but $(1,t) \sim (-1,t)$ and $(t,1) \sim (t,-1)$ for each t with $-1 \leq t \leq 1$. Let D be the quotient space $X/\sim$ equipped with the quotient topology.

a) You will probably quickly recognize D as a familiar surface, for which there are many ways to define a one-to-one continuous function of it into $\mathbb{R}^3$. Such a function is necessarily a homeomorphism onto its image for reasons we will see later, and then it is called an "embedding". Describe one such embedding. Equations are best, but a carefully explained picture is acceptable. (Note that while the square is a subset of $\mathbb{R}^2$, it seems, correctly, unlikely that D can be embedded in $\mathbb{R}^2$.)

b) In general, given a set and a quotient of it, and a metric on the set, it is not so evident how to define the quotient metric on the quotient space. (Compare with how easy it is to define the quotient topology.) But for the above space X with its metric from the Euclidean metric of the plane, it is not hard to see what the quotient metric on D should be. (Think of shortest paths, taking into account the equivalence relation.) Write a sentence or two supporting your guess as to whether there is an *isometric* embedding of D into $\mathbb{R}^3$. No proof is expected. What about an isometric embedding into $\mathbb{R}^4$?

c) Suppose we glue together parts of the boundary of X in a different way, by the equivalence relation on X given by: each point of the interior is its own equivalence class, but $(1,t) \sim (-1,-t)$ and $(t,1) \sim (t,-1)$ for each t with $-1 \leq t \leq 1$, while $(1,1) \sim (-1,1)$. Let K be the quotient space $X/\sim$ equipped with the quotient topology. This is a less familiar surface, but it may well be familiar to some of you. Write a sentence or two supporting your guess as to whether there is an embedding of

K into $\mathbb{R}^3$. (Try to draw a picture.) No proof is expected. (How else might you define this surface than as a quotient space?)

Constructing spaces by gluing (i.e. forming suitable quotient topological spaces from simple pieces) is a widely used method in geometry and algebraic topology and their many applications.

2. In mechanics (or economics, etc) the topic of "dynamics" studies how the state of a (mechanical, etc) system changes with time. The state space, i.e. the set of states, is usually modeled by a topological space, say X. For example, the state-space for a single particle moving on the line is modeled by $\mathbb{R}^2$, where the two coordinates correspond to position and momentum. Change with time is modeled by assigning to each positive real number t a function ϕ_t (assumed continuous) from X to itself that assigns to each state $x \in X$ the state to which that state has moved after the passage of t units of time (a "flow"). The basic rule that is usually satisfied is that for any $s, t \in \mathbb{R}^+$ we have $\phi_s \circ \phi_t = \phi_{s+t}$. If the system is "reversible" then negative times are permitted, and each ϕ_t is a homeomorphism of X. Then ϕ is a group homomorphism from the additive group $\mathbb{R}$ into the group of homeomorphisms of X.

The motion is often specified by an ordinary differential equation (ODE) for vector-valued functions. The ODE is said to be autonomous if it does not explicitly depend on the independent variable, which we assume here. For simplicity we take here the state space to be $\mathbb{R}^n$. The ODE is then specified by a continuous function F from $\mathbb{R}^n$ to itself (a vector field on $\mathbb{R}^n$). For any given $x \in \mathbb{R}^n$ (the "initial condition") the ODE is

$$(d\phi_t(x)/dt)(t) = F(\phi_t(x))$$

Under mild (Lipschitz) conditions the solutions are unique for any given initial condition. The existence of global solutions (i.e. defined for all times t) requires stronger conditions, but let us assume here that this holds too.

Assume that the properties indicated above hold, and that the system is reversible. For any point $x \in X$ define its "orbit" under ϕ to be the subset $\{\phi_t(x) : t \in \mathbb{R}\}$ of X. Check that being in the same orbit is an equivalence relation on X. Let O_ϕ be the set of all the orbits for ϕ. There is an evident surjection from X onto O_ϕ, and so we can equip O_ϕ with the quotient topology from X. The nature of this quotient topology often reflects important properties of the dynamics of the system.

A rich source of simple reversible mathematical dynamical systems arises as follows. Let H (for "Hamiltonian") be a real $n \times n$ matrix. It determines a vector field F by $F(x) = Hx$. Then the solution to the corresponding ODE is given by $\phi_t(x) = e^{tH}x$ (where e^{tH} is defined by the usual power-series for e^r but using matrix multiplication, so not entry-wise exponentiation in general, and convergence is for the operator norm on matrices).

For each of the matrices $\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$ and $\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$ as H, determine the corresponding set of orbits in $\mathbb{R}^2$, and the quotient topology on the orbit space. One good way to describe the topology is by parametrizing the orbits, e.g. by choosing in an attractive way one point (a "representative") in each orbit, so that there is a bijection (i.e. one-to-one and onto function) from the set of orbits to this set of representatives, and then describing the topology on the set of representatives such that the bijection is a homeomorphism. Draw an informative picture.

Discuss briefly interesting features of the quotient topology that you see, such as closures of interesting subsets of the orbit space, whether points are closed, interesting limits of sequences of orbits. In particular, determine for each case whether the quotient topology is the topology determined by a metric on the space of orbits.

If you remove the origin, to obtain the "punctured plane", then the orbits there are the "leaves" of a "one-dimensional foliation" of the punctured plane, i.e. express it as a disjoint union of curves, with each point having a neighborhood that is foliated in the "trivial" way. So from the two above matrices for H you obtain two different foliations, which are not equivalent in the sense that there is no homeomorphism that carries one foliation to the other. Foliations, including ones with higher-dimensional "leaves", occur in many situations in topology and geometry and their many applications.

If instead you use only the open upper half-plane, and its intersections with the orbits as leaves, you obtain two foliations of the upper half-plane, that are not equivalent. The upper half-plane is homeomorphic to the whole plane, and to the open unit disk, so you obtain foliations of these too. You can entertain yourselves by drawing these two foliations on the open unit disk, and then drawing more complicated foliations of the open unit disk and seeing what the quotient topologies are on the sets of leaves of the foliations.