M. Rieffel Math 202A, Problem Set 2 due September 7, 2018

Solutions for the following problems appear in several texts, and even to some extent on Wikipedia. I strongly recommend that you do these problems without consulting such sources. You will develop a stronger understanding that way. These problems are not difficult.

1. For some classes of examples it is convenient to define a topology by specifying the closure of each subset. (See the next problem). Thus a "closure operation" on a set X will in particular be a function, say cl , from the collection of all subsets of X into itself. Kuratowski gave the following axioms on such a function for it to be a closure operation, i. e. for it to actually determine a topology:

1. $\mathrm{cl}(\emptyset) = \emptyset$.
2. For every subset A of X we have $A \subseteq \mathrm{cl}(A)$.
3. For every subset A of X we have $\mathrm{cl}(\mathrm{cl}(A)) = \mathrm{cl}(A)$.
4. For any two subsets A and B of X we have $\mathrm{cl}(A) \cup \mathrm{cl}(B) = \mathrm{cl}(A \cup B)$.

a) Prove that if $\mathcal{T}$ is a topology on a set X, and if one defines cl by $\mathrm{cl}(A) = \bar{A}$ for all $A \subseteq X$, where $\bar{A}$ means the closure of A for the topology, then cl satisfies Kuratowski's axioms.

b) Prove conversely that if cl is a closure operation on a set X, and if one takes the closed subsets of X to be all those such that $\mathrm{cl}(A) = A$, then these closed subsets determine a topology on X, in the sense that their complements form the open subsets for a topology.

c) Prove that for the topology obtained in part b) the closure of every set A is $\mathrm{cl}(A)$.

2. a) (If you don't know what a ring (or at least a commutative ring) and an ideal are, then say so, and read this problem, but don't do part a) of it. However you should still be able to do part b) with the information given there.) Let R be a (unital) ring. A two-sided ideal I is said to be a "prime ideal" if $I \neq R$ and if I has the property that whenever J and K are two-sided ideals of R such that $JK \subseteq I$ (where $JK = \{ab : a \in J, b \in K\}$) then at least one of J and K is contained in I. (Note that $\{0\}$ might be prime.) Denote the set of all prime ideals of R by $\mathrm{Prime}(R)$. (When the ring is commutative, $\mathrm{Prime}(R)$ is usually called the "spectrum" of the ring, and is denoted by Spec(R).) Define a topology (the "Jacobson topology" or "hull-kernel" topology, or for commutative rings the "Zariski topology") on $\mathrm{Prime}(R)$ as follows. Given a subset A of $\mathrm{Prime}(R)$, its closure is the set of all prime ideals (the "hull") that contain the ideal that is the intersection of

all the ideals in A (the "kernel" of A). (The intersection of the prime ideals in the empty set of prime ideals can be considered to be R itself, so that the closure of the empty set is the empty set.) Prove that this definition of closure satisfies the axioms of Kuratowski, and so defines a topology by problem 1. (Note exactly where you use the fact that the ideals that you are dealing with are prime.)

b) For the ring of integers $\mathbb{Z}$ (for which the prime ideals are exactly the subsets of form $p\mathbb{Z}$ for p a prime integer, together with $\{0\}$) determine the Jacobson topology on Prime($\mathbb{Z}$) (i.e. determine its collection of open subsets). (This topology is not very useful for this simple example, but the Jacobson topology or Zariski topology is quite useful for many more-complicated rings, even rings of operators on Hilbert spaces that are related to quantum physics.)

c) Show that there is no metric on Prime($\mathbb{Z}$) that determines its Jacobson topology, i.e. is such that the topology from the metric is the same as the Jacobson topology.