M. Rieffel Math 202A, Homework 13
due Monday November 26, 2018

1. a) Prove that any continuous function from the interval $[0, 1]$ into any Banach space is integrable for Lebesgue measure on the interval.

b) Describe a natural class of topological spaces $X$ and suitable Borel measures on them for which you can prove the same result as in part a) but with $[0, 1]$ replaced by $X$ and with a similar proof. (I am not asking you to find the most general class of spaces and measures.)

2. Let $B = L^1(\mathbb{R})$ for Lebesgue measure (i.e. consisting of equivalence classes of real-valued functions). Define a $B$-valued function $f$ on the interval [0, 1] by $f(t) = \chi_{[t,t+1]}$. (By which one really means the a.e. equivalence class of $\chi_{[t,t+1]}$ in $B$.) Use the theory developed up to now in the course to:

a) Show that $f$ is a continuous function (for the usual topology on [0, 1] and the topology from the metric from the norm of $B$). Thus by problem 1 above you see that $f$ is Lebesgue integrable.

b) Calculate $\int_{[0,1]} f(t)\, dt$ for Lebesgue measure on [0, 1] (i.e. determine what function in $B$ this integral is equal to). (Do not use results about integration that you may have seen in other courses, e.g. the Riemann integral.)

3. Find a sequence $\{f_n\}$ of Lebesgue-integrable $\mathbb{R}$-valued functions on the interval $[0, 1]$ that converges a.u. (and so in measure) to an integrable function $f$ but such that the sequence $\int_{[0,1]} f_n$ does not converge to $\int_{[0,1]} f$.