M. Rieffel Math 202A, Homework 12 due November 16, 2018

For all of these problems assume that $(X, \mathcal{S}, \mu)$ is a measure space and B is a Banach space, and that all the functions mentioned below are defined on X and are $\mathcal{S}$-measurable.

1. Give an example of a sequence of $\mathbb{R}$-valued functions that converges μ-almost everywhere, but does not converge in measure.

2. Give an example of a totally finite measure space $(X, \mathcal{S}, \mu)$ and a sequence $\{f_n\}$ of $\mathbb{R}$-valued ISF's on X that is Cauchy in measure but such that the sequence $(\int f_n d\mu)$ converges to $+\infty$. Can your sequence be mean Cauchy? Why?

3. Let $\{f_n\}$ and $\{g_n\}$ be sequences of B-valued functions that converge in measure to functions f and g respectively. Prove that for any $\alpha \in \mathbb{R}$ the sequence $\{\alpha f_n + g_n\}$ converges in measure to $\alpha f + g$, and that the sequence $\{x \mapsto \|f_n(x)\|\}$ converges in measure to the function $x \mapsto \|f(x)\|$.

4. Prove that if f is a B-valued function, and if $E \in \mathcal{S}$ with $\mu(E) < \infty$, then for any $\varepsilon > 0$ there is an $F \subseteq E$ and a constant K such that $\mu(E \setminus F) < \varepsilon$ and $\|f(x)\| \leq K$ for all $x \in F$.

5. Let $\{f_n\}$ be a sequence of $\mathbb{R}$-valued functions and let $\{g_n\}$ be a sequence of B-valued functions, so that the pointwise products $f_n g_n$ are defined.

a) Prove that if $\{f_n\}$ converges in measure to the constant function 0 and if $\{g_n\}$ converges in measure to the constant function 0_B, then the sequence $\{f_n g_n\}$ converges in measure to the constant function 0_B.

b) Suppose now that $\{g_n\}$ converges in measure to a function g, and that f is an $\mathbb{R}$-valued function. Let $E \in \mathcal{S}$ with $\mu(E) < \infty$. Prove that the sequence $\{f g_n\}$ converges in measure on E to fg.

c) Indicate how a very similar argument shows that if $\{f_n\}$ is a sequence of $\mathbb{R}$-valued functions that converges to a function f, and if g is a B-valued function, then the sequence $\{f_n g\}$ converges in measure on E to fg.

d) Show by example that if the condition that $\mu(E) < \infty$ is omitted in parts b) and c) then their conclusions can fail.

e) Assume now that the sequence $\{f_n\}$ converges in measure to a function f and that the sequence $\{g_n\}$ converges in measure to a function g. Let $E \in \mathcal{S}$ with $\mu(E) < \infty$. Prove that the sequence $\{f_n g_n\}$ converges in measure on E to fg. (Hint: Apply part a) to the sequence $\{(f - f_n)(g - g_n)\}$ and use parts b) and c)).