M. Rieffel   Math 202A, Homework 11   due November 9, 2018

1. For this problem, use Lebesgue measure on the unit interval [0, 1].

a) Give an example of a sequence of continuous $\mathbb{R}$-valued functions on [0, 1] that converges almost uniformly to a discontinuous function, but does not converge uniformly.

b) Give an example of a sequence of continuous $\mathbb{R}$-valued functions on $[0, 1]$ that converges almost uniformly to a continuous function, but does not converge uniformly.

2. For this problem, use Lebesgue measure on $\mathbb{R}$. Give an example of a sequence of continuous $\mathbb{R}$-valued functions on $\mathbb{R}$ that converges pointwise to a continuous function, but does not converge almost uniformly. (So Egoroff's theorem is not true for sets whose measure is $+\infty$.)

3. Let $X = \mathbb{R}$, let S be the usual σ-field of Borel subsets of X, and let μ be the usual Lebesgue measure (restricted to S). Let V denote the vector space of equivalence classes of $\mathbb{R}$-valued ISF's (integrable simple functions) for the measure space (X, S, μ), where two ISF's are equivalent if they differ only on a null set. Equip V with the norm

$$\|f\|_1 = \int |f(x)| \, d\mu(x).$$

Give an explicit example of a sequence of elements in V that is Cauchy for this norm, but does not converge to any element of V for this norm. (Prove that your example works.) Thus, for this measure space, V is not complete for this norm.

4. Let $W = C([0, 1])$, the vector space of continuous functions on the unit interval. Let $\| \cdot \|_1$ be the familiar norm on W (defined much as in problem 3, though the Riemann integral can be used if desired). For each $t \in [0, 1]$ let ϕ_t denote the linear functional on W defined by $\phi_t(f) = f(t)$. Prove that for each t the functional ϕ_t is not continuous for the norm $\| \cdot \|_1$. (Thus you can not expect these functionals to extend to the completion of W for this norm, so for elements of the completion you can not consider their values at points of $[0, 1]$.)

5. Let V and W be as in the two previous problems.

a) For each f in W construct a sequence of ISF's which is Cauchy for the norm $\| \cdot \|_1$ and which converges pointwise to f. (Thus one can expect that the elements of W will represent elements in the completion of V.)

b) For any interval $[a, b)$ in $[0, 1]$ let f be its characteristic function. Construct a sequence of elements of W that is Cauchy for the norm $\| \cdot \|_1$ and which converges pointwise to f. (Thus one can expect that each ISF represents an element in the completion of W. Thus one expects that the completions of V and W coincide.)