M. Rieffel Math 202A Homework 1 due August 31, 2018

Justify your answers to the problems below.

1. For each of the following parts, find a pair of metric spaces, (X, d_X), (Y, d_Y) and a function $f : X \to Y$ that satisfies the given conditions. Actually, it is possible to always use $\mathbb{R}$ with its usual metric.

a) f is injective (i.e. one-to-one) and Lipschitz, but not isometric.

b) f is uniformly continuous but not Lipschitz.

c) f is continuous but not uniformly continuous.

2. Let $\mathbb{Q}$ denote the set of rational numbers, viewed as a dense subset of $\mathbb{R}$, equipped with the the metric that is the restriction of the usual metric on $\mathbb{R}$.

a) Give an example of a continuous function, f, from $\mathbb{Q}$ to $\mathbb{R}$ that does not have a continuous extension to a function from $\mathbb{R}$ to $\mathbb{R}$.

b) For your answer, f, to part a) there necessarily exists a Cauchy sequence $\{q_n\}$ in $\mathbb{Q}$ such that $\{f(q_n)\}$ is not a Cauchy sequence. Explicitly exhibit such a Cauchy sequence.

c) Does there exist a continuous function, f, from $\mathbb{R}$ to $\mathbb{R}$ for which there exists a Cauchy sequence $\{r_n\}$ in $\mathbb{R}$ such that $\{f(r_n)\}$ is not a Cauchy sequence?

3. Let (X, d_X) and (Y, d_Y) be metric spaces, and let $(j_X, \bar{X})$ and $(j_Y, \bar{Y})$ be completions for them (where j_X is an isometry from X to $\bar{X}$ with dense range, and similarly for Y).

a) Show that if f is a uniformly continuous function from X to Y then there is a uniformly continuous function, $\bar{f}$, from $\bar{X}$ to $\bar{Y}$ that extends f in the evident sense. Make clear you know what the evident sense is. Show that $\bar{f}$ is unique.

b) Let (Z, d_Z) be another metric space, and let g be a uniformly continuous function from Y to Z. Let $(j_Z, \bar{Z})$ be a completion of Z. For notation as in part a), prove that the extension of $g \circ f$ to a function from $\bar{X}$ to $\bar{Z}$ agrees with $\bar{g} \circ \bar{f}$.

[One specifies a category by specifying what the objects are, and what the allowed functions (perhaps in a generalized sense, "morphisms") are. The main requirement is that the composition of any two allowed functions must be an allowed function. For each object there also must be an allowed "identity" function from the object to itself. A morphism from one category to another is called a "functor". It must carry objects to objects and functions to functions, respecting composition of functions. Thus part b) above is the main step in showing that the process of assigning to each metric space its completion, and to each uniformly continuous function its extension to the completions, is essentially a functor from the category of metric spaces and uniformly continuous functions to the category of complete metric spaces and uniformly continuous functions. To make this more precisely correct one should use a consistent method of assigning completions, such as always using the method of equivalence classes of Cauchy sequences. Many “universal constructions” in mathematics can profitably be seen to be functors. Be alert for them.]