Solutions to hw 5.

Note Title 7/26/2009

1. (BC, p.178, 1) $g(z) = e^{f(z)}$,

$|g(z)| = \exp(\operatorname{Re}(f(z)))$

since $\operatorname{Re}(f(z)) = u(x,y)$ is bounded, the function $|g(z)|$ is also bounded; because $f(z)$ is entire and e^z is entire, g is also entire, and $\Rightarrow$ by the Liouville's theorem $g(z) = \text{const}$.

(BC, p. 179, 2),

$P(z) = a_0 + a_1 z + \dots + a_{n-1} z^{n-1} + a_n z^n$

$|P(z)| \le |a_0 + \dots + a_{n-1} z^{n-1}| + |a_n||z|^n$,

$$\frac{|a_0 + \dots + a_{n-1} z^{n-1}|}{|a_n||z|^n} = \left| \frac{a_0}{|a_n||z|^n} + \dots + \frac{a_{n-1}}{|a_n||z|} \right|$$

for each $f_k = \frac{a_k}{|a_n||z|^k}$ we can choose R_k s.t. $|f_k| < \frac{1}{n}$ for each $|z| > R_k$ (because $f_k \to 0$ as $z \to \infty$) choose $R = \max_k (R_k)$. For all $|z| > R$ we will have

$$\frac{|a_0 + \dots + a_{n-1} z^{n-1}|}{|a_n| |z|^n} < 1$$

$\Rightarrow |P(z)| < 2|a_n||z|^n$.

(BC, p.178, 3) Apply the Corollary on p.178 for $g(z) = \frac{1}{f(z)}$.

(BC, p.178, 6) Consider $g(z) = e^{f(z)}$

f is continuous on $D \Rightarrow g$ is also continuous (comp. of cont. fncns)

It is also analytic in the interiour of $D \Rightarrow$ by the Max. modulus thm and its corollary $|g(z)| = e^{\operatorname{Re} f(z)}$ has no maximum in the interiour, but reaches the maximum at the boundary of D. Since e^x is monotonic, the same is true for $\operatorname{Re} f(z)$.

2. f is an analytic function in a disc $\dot{D}_\delta = \{0 < |z| < \delta\}$ and $f(z) \to \infty$ as $z \to 0$, then f has a pole at $z = 0$.

$\blacksquare$ f can not have a sequence of zeroes z_n ($f(z_n) = 0$) s.t. $z_n \to 0$ (since $f(z) \to \infty$ as $z \to 0$). Choose ε s.t. $f(z)$ does not vanish in $\dot{D}_\varepsilon$

Consider $g(z) = \frac{1}{f(z)}$ on $\dot{D}_\varepsilon$. It is analytic, and $g(0) = 0$. $\Rightarrow$ $g(z)$ is equal to its Taylor series about $z = 0$ in some nbd. of $z = 0$.

$$g(z) = \sum_{n \ge 1} z^n \frac{g^{(n)}(0)}{n!} \quad (\text{since } g(0) = 0)$$

Denote by k the degree of the first non-zero monomial in this power series.

$$f(z) = \frac{1}{g(z)} = \frac{1}{z^k \left(\frac{g^{(k)}(0)}{k!} + \dots \right)}$$

has a pole of order k. $\blacksquare$

3. $0 < |z| < 1$, $z = re^{i\varphi}$, $0 < r < 1$, $0 \le \varphi < 2\pi$

$$e^{-\frac{1}{z}} = e^{-\frac{\cos\varphi}{r}} e^{i \frac{\sin\varphi}{r}}$$

1) $|f| = e^{-\frac{\cos\varphi}{r}} = \begin{cases} 0<|f|\le 1, & 0\le\cos\varphi\le 1 \\ 1<|f|<+\infty, & -1\le\cos\varphi<0 \end{cases}$

$0<r<1$

2) Let $\frac{\cos\varphi}{r} = t$, $\quad r = \frac{\cos\varphi}{t}$, $\quad \frac{\sin\varphi}{r} = t\tan(\varphi)$

$(-\infty<t<+\infty)$

a. $0\le t$, $\quad \cos\varphi\ge 0$, $\quad -\infty<\frac{\sin\varphi}{r}<+\infty$

b. $t>0$, $\quad \cos\varphi<0$, $\quad -\infty<\frac{\sin\varphi}{r}<+\infty$

$\Rightarrow$ for fixed $|f|$, $\operatorname{Arg}(f)$ takes all possible values $\Rightarrow$

range$(f) = \mathbb{C}\setminus\{f=0\}$

4. $\frac{1}{\sin z}$ has poles of order 1 at $z=\pi n$, $n\in\mathbb{Z}$

5. $\frac{\sin(\frac{1}{z})}{(z-1)^2(z^2+1)}$ has an essent. sing. at $z=0$

simple poles (i.e. poles of order 1) at $z=\pm i$,

pole of order 2 at $z=1$.

6. (BC, p.205–206)

Answers are in the textbook.

7. $f = \frac{1}{z^2-z^4} = \frac{1}{z^2(1-z)(1+z)}$ has singular points: $\begin{cases} z=0, & \text{pole of order 2} \\ z=\pm 1, & \text{poles of order 1} \end{cases}$

Step 1. $\frac{1}{(1-z)(1+z)} = \frac{1}{2}\frac{1}{1-z} + \frac{1}{2}\frac{1}{1+z}$

a) $z=0$,

$\frac{1}{z^2(1-z)} = \sum_{n=0}^{\infty} z^{n-2}$, $\quad \frac{1}{z^2(1-z)} = \sum_{n=0}^{\infty}(-1)^n z^{n-2}$

$f = \frac{1}{2z^2(1-z)} + \frac{1}{2z^2(1+z)} = \frac{1}{2}\sum_{n=0}^{\infty}\left(1+(-1)^n\right)z^{n-2}$

$= \sum_{\substack{n=0\\ n-\text{even}}}^{\infty} z^{n-2} = \sum_{k=0}^{\infty} z^{2k-2} = \frac{1}{z^2} + 1 + z^2 + \dots$

b) $z=1$, (i) $\frac{1}{z^2(1-z)} = \frac{1}{(1-z)(1-(1-z))^2} =$

$\left\langle \frac{1}{(1-t)^2} = \frac{d}{dt}\frac{1}{1-t} = \sum_{n=1}^{\infty} n t^{n-1} = \sum_{n=0}^{\infty}(n+1)t^n \right\rangle$

$= \frac{1}{1-z}\sum_{n=0}^{\infty}(n+1)(1-z)^n = \sum_{n=0}^{\infty}(n+1)(1-z)^{n-1} =$

$= \sum_{m=-1}^{\infty}(m+2)(1-z)^m$,

(ii) $\frac{1}{z^2(1+z)} = \frac{1}{z^2} - \frac{1}{z} + \frac{1}{1+z} =$

$= \frac{1}{(1-(1-z))^2} - \frac{1}{1-(1-z)} + \frac{1}{2(1-\frac{1-z}{2})}$

$= \sum_{n=0}^{\infty}(n+1)(1-z)^n - \sum_{n=0}^{\infty}(1-z)^n + \sum_{n=0}^{\infty}\left(\frac{1}{2}\right)^{n+1}(1-z)^n$

$= \sum_{n=0}^{\infty}\left(n+2^{-n-1}\right)(1-z)^n$,

(iii) $\frac{1}{2}(i) + \frac{1}{2}(ii) = \frac{1}{1-z} + \sum_{n=0}^{\infty}\left(2n+2+2^{-n-1}\right)(1-z)^n$

is the Laurent expansion for $f(z)$ about $z=-1$

3. $z=-1$, $\quad f(z) = f(-z)$

$$\Rightarrow f(z) = \frac{1}{1+z} + \sum_{n=0}^{\infty} \left(2n+2+2^{-n-1}\right)(1+z)^n$$

is the Laurent series for $f(z)$ about $z=-1$.

8. Partial fraction decompositions:

(i) $$\frac{1}{(z-1)^2(z^4-1)} = \frac{1}{(z-1)^3(z+1)(z+i)(z-i)}$$

$$\frac{1}{(z+1)(z+i)(z-i)} = \frac{1}{(z+1)2} + \frac{1}{(z-i)2i(1+i)} - \frac{1}{(z+i)2i(1-i)},$$

$$\frac{1}{(z-1)^3(z+1)} = \frac{1}{(z-1)^3 2} - \frac{1}{(z-1)^2 4} + \frac{1}{(z-1)4} - \frac{1}{8(z+1)}$$

$$\frac{1}{(z-1)^3(z+i)} = \frac{1}{(z-1)^3(1+i)} - \frac{1}{(z-1)^2(1+i)^2} + \frac{1}{(z-1)(1+i)^3} - \frac{1}{(1+i)^3(z+i)},$$

Similarly

$$\frac{1}{(z-1)^3(z-i)} = \frac{1}{(z-1)^3(1-i)} - \frac{1}{(z-1)^2(1-i)^2} + \frac{1}{(z-1)(1-i)^3} - \frac{1}{(1-i)^3(z-i)},$$

From here:

$$\frac{1}{(z-1)^2(z^4-1)} = \frac{1}{4(z-1)^3} - \frac{1}{8(z-1)^2} + \frac{1}{8(z-1)} + \left\{-\frac{1}{8(z+1)} + \frac{1}{2i(1+i)^2(z-1)^3} - \frac{1}{2i(1+i)^3(z-1)^2} + \frac{1}{2i(1+i)^4(z-1)} - \frac{1}{2i(1+i)^4(z+i)}\right\} + \left\{\dots\right\}$$

$$\frac{z}{(z-1)(z+i)(z-i)} = \frac{1}{(z-1)2} + \frac{i}{(z-i)2i(i+1)} - \frac{i}{(z-i)2i(1-i)},$$

9. a) $f = \dfrac{1}{z^2+z^4} = \dfrac{1}{z^2}\left(1 - z^2 + z^4 + \dots\right)$

- $z=0$ pole of order 2
  $\operatorname{Res}_{z=0}(f) = 0$
- $z=i$, $f = \dfrac{1}{z^2(z+i)(z-i)}$,
  pole of order 1, $\operatorname{Res}_{z=i}(f) = -\dfrac{1}{2i} = \dfrac{i}{2}$
- $z=-i$, pole of order 1,
  $\operatorname{Res}_{z=-i}(f) = -\dfrac{i}{2}$

b. $f = \dfrac{e^{1/z}}{1-z}$,

- $z=0$, essential singularity

$$f = \left(1 + \frac{z^{-1}}{1!} + \frac{z^{-2}}{2!} + \dots + \frac{z^{-n}}{n!} + \dots\right)\left(1 + z + z^2 + \dots\right)$$

$$= z^{-1}\left(\frac{1}{1!} + \frac{1}{2!} + \frac{1}{3!} + \dots\right) + \dots = \frac{e-1}{z} + \dots$$

$\operatorname{Res}_{z=0}(f) = e-1$,

- $z=1$, simple pole (pole of order 1)
  $\operatorname{Res}_{z=1}(f) = e$

c. $\dfrac{1}{\sin z}$, simple poles $z = \pi n$

$$\operatorname{Res}_{z=\pi n}(f) = \frac{1}{\cos(\pi n)} = (-1)^n$$

d. $f = \sin\left(\frac{1}{z}\right)$,

essential singularity at $z = 0$.

$$\sin\left(\frac{1}{z}\right) = \frac{1}{z} - \frac{1}{z^3}\frac{1}{3!} + \cdots + \frac{(-1)^n z^{-2n-1}}{(2n+1)!} + \cdots$$

$$\operatorname*{Res}_{z=0}(f) = 1,$$