Solutions, hw 4

Note Title 7/21/2009

1. The function

$$f(z) = \operatorname{Re}\cos z + \operatorname{Im}\cos z =$$
$$= \operatorname{Re}(\cos(x+iy)) + \operatorname{Im}(\cos(x+iy)) =$$
$$= \operatorname{ch} x \cos y + \sin y \operatorname{sh} x$$

is not analytic: it does not satisfy the Cauchy–Rieman equations.

2. $u_x = v_y$, $u_y = -v_x$

$0 = 0$, $u'(y) = -v'(x)$,

$\Rightarrow u'(y) = -v'(x) = D$ - real constant

$\Rightarrow u(y) = Dy + A$, $v(x) = -Dx + B$

$f(x,y) = Dy + A - iDx + iB = -iDz + (A+iB)$

3. The function $f(z)$ is entire.

Indeed, consider

$$g(z) = 1 - \frac{z^2}{3!} + \frac{z^4}{5!} + \dots = \sum_{n=0}^{\infty} \frac{(-1)^{n+1}}{(2n+1)!} z^{2n}$$

1. $g(z)$ is entire ($R = \infty$)
2. $g(z) = \frac{\sin z}{z}$ when $z \neq 0$
3. $g(0) = 1$

$\Rightarrow f(z) = g(z) \Rightarrow f$ is entire.

4. $f(z)$ is analytic outside $z^5 = 1$ and $z^3 = -1$, i.e. outside of

$z = e^{\frac{2\pi i}{5} + \frac{2\pi i}{5} n}$, $n = 0,1,2,3,4$,

$e^{i\frac{\pi}{3} + \frac{2\pi i}{3} n}$, $n = 0,1,2$,

5. (BC, p.160-163)

(c) $f(z)$ is analytic outside solutions to $z^2 + 2z + 2 = 0$, $z_{1,2} = \frac{2 \pm \sqrt{4 - 4\cdot 2}}{2}$

$= 1 \pm i$, $|z_{1,2}| = \sqrt{2} > 1$

$f(z)$ is analytic inside $|z| = 1$

$\Rightarrow \int_{|z|=1} f\, dz = 0$

(d) $\operatorname{Log}(z+2)$ has a branch cut:

$\operatorname{Log}(z+2)$ is analytic inside $|z| = 1$ $\Rightarrow$

$\Rightarrow \int_{|z|} f\, dz = 0$

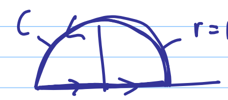
(e) $\tan z$ is analytic outside points where $\cos z = 0$

i.e. when $z \neq \frac{\pi}{2} + \pi n$

there are no such points inside $|z| = 1$

$\Rightarrow f$ is analytic there $\Rightarrow \int_{|z|=1} f\, dz = 0$

6.



$f(z) = \sqrt{r}\, e^{i\frac{\theta}{2}}$

$0 \le r \le 1$, $0 \le \theta \le \pi$

$$\int_0^{\pi} r e^{i\frac{\theta}{2}} d(re^{i\theta})\Big|_{r=1} = i \int_0^{\pi} e^{i\frac{3}{2}\theta} d\theta =$$

$$= i \frac{2}{3i}\left(e^{\frac{3}{2}i\pi} - 1\right) = -\frac{2}{3}(1+i)$$

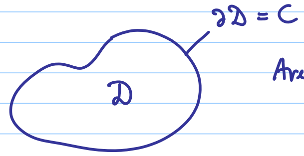
$$\int_{-1}^{0} f\, dz = -\int_0^1 \sqrt{-x}(-dx) = i\frac{2}{3}$$

$$\int_0^1 f\, dz = \frac{2}{3}$$

$$\int_C f\, dz = 0$$

f is not analytic on C (branch cut from 0 to $-\infty$).

⑦



$$\text{Area} = \iint_D dx\,dy$$

$$-i\int_C \bar{z}\,dz = \int_C (x\,dy - y\,dx) - i\int_C (x\,dx + y\,dy)$$

Green's theorem:

$$\int_C (P\,dx + Q\,dy) = \iint_D (Q_x - P_y)\,dx\,dy$$

$$\Rightarrow \int_C (x\,dx + y\,dy) = 0$$

$$\int_C (x\,dy - y\,dx) = \iint_D 2\,dx\,dy$$

$$\Rightarrow \frac{1}{2i}\int_C \bar{z}\,dz = \text{Area}.$$

6. (BC, p. 170–172)

③ $C = \{z \mid |z| = 3\}$

if $|z| > 3$ the point z is outside of $\mathrm{Int}(C) \Rightarrow \displaystyle\int_C \frac{f(s)}{s-z}\,ds = 0$

for an entire fncn f, $f(s) = 2s^2 - s - 2$ is a polynomial and $\Rightarrow$ is entire.

⑥ In the proof of the theorem on page 164 and of the formula (5) p. 167 we used only continuity of $f \Rightarrow$ ⑥.

7. (BC, p. 195–197)

② trivial

⑬ $$\frac{1}{4z - z^2} = \frac{1}{4z}\cdot\frac{1}{1-\frac{z}{4}} =$$

$$= \frac{1}{4z}\sum_{n=0}^{\infty}\frac{z^n}{4^n} = \frac{1}{4z} + \sum_{n=1}^{\infty}\frac{z^{n-1}}{4^{n+1}}$$

$$= \frac{1}{4z} + \sum_{k=0}^{\infty}\frac{z^k}{4^{k+2}}, \qquad n = k+1,$$