Solutions, week 3.

Note Title 7/14/2009

1. $\sum_{n=1}^{\infty} n z^n$ radius of conv. $= 1$

$\sum_{n=1}^{\infty} (-2)^n z^n$, radius of conv. is $\frac{1}{2}$

$\Rightarrow \sum_{n=1}^{\infty} (n + (-2)^n) z^n$ has radius of conv $\min(1, \frac{1}{2}) = \frac{1}{2}$.

2. $\sum_{n=1}^{\infty} \frac{(-1)^n z^n}{\sqrt{n}}$ has radius of convergence 1

3. $\sum_{n=1}^{\infty} \frac{1}{n^{3/2}} z^n$, $R = 1$

$\sum_{n=1}^{\infty} n^2 z^n$

1. $R = 1$, divergent when $|z| = 1$ (div. test)
$\Rightarrow$ the region of convergence is $|z| < 1$

2. $\sum_{n=0}^{\infty} z^n = \frac{1}{1-z}$, $\overset{\frac{d}{dz}}{\Rightarrow}$ $\sum_{n=1}^{\infty} n z^{n-1} = \frac{1}{(1-z)^2}$

(can differentiat inside the region of conv.) $\Rightarrow \sum_{n=1}^{\infty} n z^n = \frac{z}{(1-z)^2}$

differentiate again, $\frac{d}{dz}\left(\frac{z}{(1-z)^2}\right) = \frac{1}{(1-z)^3}$

$\Rightarrow \sum_{n=1}^{\infty} n^2 z^{n-1} = \frac{1}{(1-z)^3}$

$\Rightarrow \boxed{\sum_{n=1}^{\infty} n^2 z^n = \frac{z}{(1-z)^3}}$

3. The function is defined for all $z \neq 1$ i.e. on $\mathbb{C} \setminus \{1\}$.

---

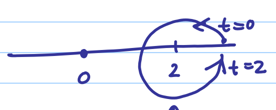
integrals:

1. $\int_0^1 (t^4 + i t^2)\, dt = \frac{1}{5} + \frac{i}{3}$

2. $\int_0^{2\pi} \frac{\cos t - i \sin t}{\sin t + i \cos t}\, dt = \int_0^{2\pi} (-i)\, dt = -i 2\pi$

3. $\int_0^{2\pi} \frac{-\sin t + i \cos t}{\cos t + i \sin t + 2}\, dt = \int_0^{2\pi} \frac{i e^{it}\, dt}{e^{it} + 2} =$

$= \ln(e^{it} + 2)\Big|_0^{2\pi} = 0$



$\mathrm{Arg}(e^{it}+2)$ does not change as t goes from 0 to 2π along C $\Rightarrow$

4. $\frac{1}{z(z+1)(z+2)} = \frac{1}{2z} - \frac{1}{z+1} - \frac{1}{2(z+2)}$,

$\int_1^T \frac{dz}{z(z+1)(z+2)} = \ln\left(\frac{\sqrt{z(z+2)}}{z+1}\right)\Big|_1^T =$

$= \ln\left(\frac{\sqrt{T(T+2)}}{T+1}\right) - \ln\left(\frac{\sqrt{3}}{2}\right)$

$\swarrow T \to \infty$

0

$\Rightarrow \int_1^{\infty} \frac{dz}{z(z+1)(z+2)} = \ln\left(\frac{2}{\sqrt{3}}\right)$