Solutions, week 2

Note Title 7/14/2009

1. $u^2+v^2=C \Rightarrow 2uu_x+2vv_x=0$

$2uu_y+2vv_y=0$

Cauchy - Riemann: $u_x=v_y$, $u_y=-v_x$

$\Rightarrow vv_x+uv_y=0$, $vv_y-uv_x=0$

- if $v_x, v_y \neq 0$ these equations should be dependent $\Rightarrow v^2+u^2=0$ (det $=0$)
  $\Rightarrow u=v=0$ contradiction
- $v_x=v_y=0 \Rightarrow v=$ const. $\Rightarrow$ by C-R
  $u_x=u_y=0 \Rightarrow u=$ const.

$\Rightarrow f = u+iv = $ const. $\in \mathbb{C}$

2. Cauchy - Riemann:

$v_y = u_x = 2x+a$, $v_x=-u_y=2y$

$v=(2x+a)y+f(x)$, $v=2xy+g(y)$

$v=2xy+ay+c$

3. $f_x = \frac{1}{z^2}e^{-\frac{1}{z}}$, $f_y = \frac{i}{z^2}e^{-\frac{1}{z}}$

$\Rightarrow f_x=-if_y$, $\Rightarrow$ analytic for all $z\neq 0$.

4. $f_x = c_1+2c_2x+\dots+nc_nx^{n-1}+\dots$

$f_y=0$

Cauchy - Riemann: $f_x=-if_y \Rightarrow c_k=0$, $k=1,2,\dots$

$\boxed{f=c_0}$

5. The Ratio Test:

1. $\left|\frac{c_{n+1}}{c_n}\right| = \frac{(2n+1)!}{(2n+3)!} \xrightarrow[n\to\infty]{} 0$, $\Rightarrow R=\infty$

2. $\left|\frac{c_{n+1}}{c_n}\right| = \frac{(n+1)!}{(n+1)^{n+1}}\,\frac{n^n}{n!} = \frac{(n+1)n^n}{(n+1)^{n+1}} =$

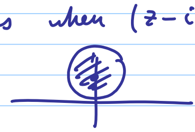
$= \frac{n^n}{(n+1)^n} = \left(1+\frac{1}{n}\right)^{-n} \xrightarrow[n\to\infty]{} e^{-1}$

$\Rightarrow R=e$

6. $\sum_{n=1}^{\infty} n w^n$, $R=1$, the series diverges for $|w|=1$ by the div. test.

$\Rightarrow$ the region of conv. $|w|<1$

$\Rightarrow \sum_{n=1}^{\infty} n(z-i)^n$ converges when $|z-i|<1$



7. $u_x=v_y$, $\underset{0}{u_y}=-\underset{0}{v_x}$ ← OK

$u'(x)=v'(y)$, possible only when

$u'(x)=v'(y)=D$ (real constant)

$\Rightarrow f = Dx+iDy+A+iB = Dz+C$

$D\in\mathbb{R}$, $C\in\mathbb{C}$

BC, p.121, 3:

$n\neq 0$ $\int_0^{2\pi} e^{in\theta}\,d\theta = \int_0^{2\pi}\cos(n\theta)\,d\theta + i\int_0^{2\pi}\sin(n\theta)\,d\theta$

$= \frac{1}{n}\sin(n\theta)\Big|_0^{2\pi} - \frac{i}{n}\cos(n\theta)\Big|_0^{2\pi} = 0$

$n=0$ $\int_0^{2\pi} e^{i0}\,d\theta = \int_0^{2\pi} d\theta = 2\pi$

$\Rightarrow \int_0^{2\pi} e^{im\theta}e^{-in\theta}\,d\theta = \begin{cases} 0, & n\neq m \\ 2\pi, & n=m \end{cases}$

BC, p. 136,

8. $\int_C z^m \bar{z}^n\,dz = \int_0^{2\pi} e^{im\theta-in\theta}\,ie^{i\theta}\,d\theta$

$= \begin{cases} 0, & n\neq m+1 \\ 2\pi i, & n=m+1 \end{cases}$

9. $$\int_C \bar{z}\, dz = \int_{-2}^{2} \left(\sqrt{4-y^2} - iy\right)\left(-\frac{y}{\sqrt{4-y^2}} + i\right) dy$$

$C = \{ z = \sqrt{4-y^2} + iy \}$
$-2 \le y \le 2$

$$= \int_{-2}^{2} \left(\sqrt{4-y^2} - iy\right)\left(-\frac{i}{\sqrt{4-y^2}}\right)\left(\sqrt{4-y^2} + iy\right) dy$$

$$= -i\int_{-2}^{2} \frac{4-y^2+y^2}{\sqrt{4-y^2}}\, dy = -4i \int_{-1}^{1} \frac{dt}{\sqrt{1-t^2}}$$

$(y = 2t)$

$$(t = \sin\theta) = -4i \int_{-\frac{\pi}{2}}^{\pi/2} \frac{\cos\theta\, d\theta}{\cos\theta} = -4i \cdot \pi =$$

$$= -4\pi i,$$

10. (a) $$\int_{C_0} f(z - z_0)\, dz = \int_0^{2\pi} f(z_0 + Re^{i\theta} - z_0)\, iRe^{i\theta} d\theta$$

$$= \int_0^{2\pi} f(Re^{i\theta})\, iRe^{i\theta} d\theta = \int_C f(z)\, dz$$

(b) $$\int_{C_0} (z - z_0)^{n-1} dz = \int_C z^{n-1} dz = i\int_{-\pi}^{\pi} R^n e^{in\theta} d\theta$$

$$= \begin{cases} 0, & n \neq 0 \\ 2\pi i, & n = 0 \end{cases}$$ (see above).