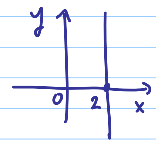
Solutions, hw 1

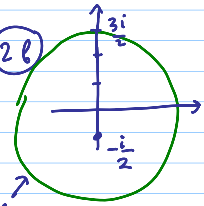
Note Title 7/21/2009

BC :

p.14 : (2a)



(2b)



circle of radius 2 centered at $-i/2$

(7) $|\mathrm{Re}(2+\bar{z}+z^3)| \le 4$, when $|z| \le 1$

$z = re^{i\varphi}$, $\mathrm{Re}(2+\bar{z}+z^3) = 2 + r\cos\varphi + r^3\cos 3\varphi$

$|2 + r\cos\varphi + r^3\cos 3\varphi| \le 2 + r + r^3 \le 2+1+1 = 4$

in the region $|z| \le 1$ (or $r \le 1$)

(14) $z = x+iy$, $z^2 + \bar{z}^2 = x^2 + 2ixy - y^2 +$
$+ x^2 - 2ixy - y^2 = 2(x^2 - y^2)$

Thus, $z^2 + \bar{z}^2 = 2$ is equivalent to
$x^2 - y^2 = 1$

p.23 (5) key identities:

$1 - \sqrt{3}\,i = 2e^{-i\frac{\pi}{3}}$, $\sqrt{3} + i = i(1 - i\sqrt{3}) = 2i e^{-i\frac{\pi}{3}}$,

$1 + \sqrt{3}\,i = 2e^{i\frac{\pi}{3}}$,

$1 \pm i = \sqrt{2}\, e^{\pm i\frac{\pi}{4}}$

(11) (a) $e^{in\theta} = \cos n\theta + i n \theta$, also

$$e^{in\theta} = (\cos\theta + i\sin\theta)^n = \sum_{k=0}^{n} \binom{n}{k} (\cos\theta)^k (i\sin\theta)^{n-k}$$

$\Rightarrow$

$$\cos n\theta + i\sin n\theta = \sum_{k=0}^{n} \binom{n}{k} (\cos\theta)^k\, i^{n-k} (\sin\theta)^{n-k}$$

Take the real part of this:

$$i^{n-k} = \begin{cases} (-1)^{\frac{n-k}{2}}, & \text{when } n-k \text{ is even} \\ \text{imaginary}, & \text{when } n-k \text{ is odd} \end{cases}$$

$$\Rightarrow \cos n\theta = \sum_{\substack{k=0 \\ n-k = \text{even}}}^{n} \binom{n}{k} \cos\theta^k \sin\theta^{n-k}$$

$$= \sum_{\ell=0}^{[n/2]} \binom{n}{2\ell} (\cos\theta)^{n-2\ell} (\sin\theta)^{2\ell},$$

$[\frac{n}{2}]$ = the integer part of $n/2$.

(b) $(\sin\theta)^{2\ell} = (\sin^2\theta)^\ell = (1-\cos^2\theta)^\ell$

$$\Rightarrow \cos n\theta = T_n(x) = \sum_{k=0}^{[n/2]} \binom{n}{2k} (-1)^k x^{n-2k} (1-x^2)^k$$

p. 30 (6) zeroes of $z^4 + 4$ are solutions to $z^4 = -4$, i.e.,

$z = \sqrt{2}\, e^{i\frac{\pi}{4}}$, $\sqrt{2}\, e^{i\frac{\pi}{4} + \frac{\pi}{2}i} = -\sqrt{2}\, e^{-i\frac{\pi}{4}}$

$\sqrt{2}\, e^{i\frac{\pi}{4} + i\pi} = -\sqrt{2}\, e^{i\frac{\pi}{4}}$, $\sqrt{2}\, e^{i\frac{\pi}{4} + \frac{3\pi i}{2}} = \sqrt{2}\, e^{-i\frac{\pi}{4}}$

(7) $c^n = 1$ and $c \ne 1$

Geometric sum :

$$1 + c + \dots + c^{n-1} = \frac{1-c^n}{1-c} = 0$$

(8) (a) trivial

(b) $z_{1,2} = \dfrac{-b \pm (b^2 - 4ac)^{1/2}}{2a} =$

$= \dfrac{-2 \pm \sqrt{4 - 4(1-i)}}{2} = -1 \pm \sqrt{i} =$

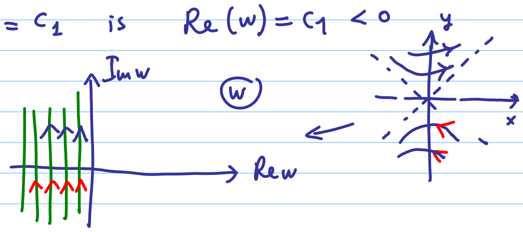
$= -1 \pm e^{i\frac{\pi}{4}} = -1 \pm \frac{\sqrt{2}}{2}(1+i)$,

p.37 (1)(b) $z \ne 0$ (d) $|z| \ne 1$

p.44

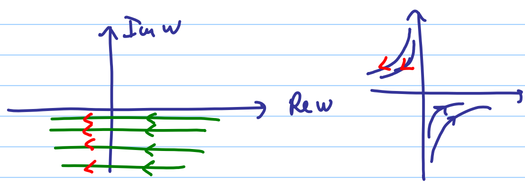
$z = x+iy, \quad z^2 = (x^2-y^2) + i\cdot 2xy$

(i) $x^2-y^2 = c_1$ is $\mathrm{Re}(w) = c_1 < 0$



$y = \pm\sqrt{x^2-c_1}$, $\mathrm{Im}\,w = \pm 2x\sqrt{x^2-c_1}$, $-\infty < x < +\infty$

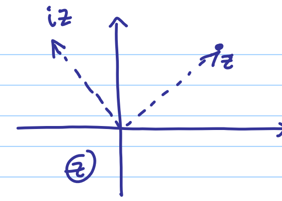
(ii) $2xy = c_2$ is $\mathrm{Im}(w) = c_2 < 0$



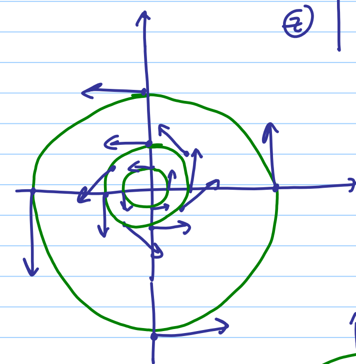
$x = \dfrac{c_2}{2y}$, $\quad \mathrm{Re}(w) = x^2-y^2 = \dfrac{c_2^2}{4y^2} - y^2$

$-\infty < y < +\infty$

8 (a) $w = iz$



$i(x+iy) = -y+ix$



(b) $w = \dfrac{z}{|z|}$

unit vectors normal to circles centered at 0.