Solutions to practice problems.

Note Title 7/28/2009

1.

(a) . $\pi z = \pm 1$, $z = \pm \frac{1}{\pi}$,

$$\sin\left(\pm \frac{1}{1/\pi}\right) = \sin(\pm\pi) = 0$$

$$\lim_{z \to \frac{1}{\pi}} \frac{\sin\left(\frac{1}{z}\right)}{(\pi z - 1)(\pi z + 1)} \underset{\text{l'Hospital}}{=} \lim_{z \to \frac{1}{\pi}} \frac{-\frac{1}{z^2}\cos\left(\frac{1}{z}\right)}{\pi(\pi z + 1)} =$$

$$= \frac{\pi^2}{2\pi} = \frac{\pi}{2},$$

$$\lim_{z \to -\frac{1}{\pi}} \frac{\sin\left(\frac{1}{z}\right)}{\pi^2 z^2 - 1} = -\frac{\pi}{2}$$

if one defines $f\left(\frac{1}{\pi}\right) = \frac{\pi}{2}$, $f\left(-\frac{1}{\pi}\right) = -\frac{\pi}{2}$

the function $f(z)$ is analytic at $\pm\frac{1}{\pi}$

- $z = 0$ is an essential singularity

(b) $$f = \frac{1}{\cos z - 1 - z^2/2},$$

(i) Obviouse singularity : $z = 0$

$$\left(\cos z - 1 - \frac{z^2}{2}\right)' = -\sin z - z \overset{z=0}{=} 0$$

$$\left(\cos^2 z - 1 - \frac{z^2}{2}\right)'' = -\cos z - 1 \overset{z=0}{=} -2 \neq 0$$

$\Rightarrow$ f has a pole of order 2 at $z = 0$

(ii) Other singularities :

$$\cos z - 1 - \frac{z^2}{2} = 0$$

no solutions for real z, other then $z = 0$.

2. (a) $$\frac{z}{1+z^4} = z \sum_{n=0}^{\infty} (-1)^n z^{4n} = \sum_{n=0}^{\infty} (-1)^n z^{4n+1}$$

(b) $$\frac{1}{(1-z)(z^2-1)} = -\frac{1}{(z-1)^2(z+1)} = -\frac{1}{2(z-1)^2\left(\frac{z-1}{2}+1\right)}$$

$$= -\frac{1}{2(z-1)^2\left(1+\frac{z-1}{2}\right)} = -\frac{1}{2(z-1)^2} \sum_{n=0}^{\infty} \frac{(-1)^n}{2^n} (z-1)^n$$

$$= -\frac{1}{2(z-1)^2} + \frac{1}{4(z-1)} + \sum_{n=0}^{\infty} \frac{(-1)^{n-1}}{2^{n+3}} (z-1)^n ,$$

(c) $$\frac{\sin z^2}{z^3} = \frac{1}{z^3} \sum_{n=0}^{\infty} (-1)^n \frac{z^{2n+1}}{(2n+1)!} =$$

$$= \frac{1}{z^2} + \sum_{n=1}^{\infty} (-1)^n \frac{z^{2(n-1)}}{(2n+1)!}$$

(d) $$\frac{\sin\left(\frac{\pi z}{2}\right)}{1+z} = \frac{\sin\left(\frac{\pi}{2}(z+1-1)\right)}{z+1} =$$

$$= -\frac{\cos\left(\frac{\pi}{2}(z+1)\right)}{z+1} = -\frac{1}{z+1} \sum_{n=0}^{\infty} (-1)^n \frac{(z+1)^{2n}}{(2n)!}$$

$$= -\frac{1}{z+1} - \sum_{n=1}^{\infty} (-1)^n \frac{(z+1)^{2n-1}}{(2n)!} ,$$

3. (a) $(z+1)\cos\left(\frac{1}{z}\right)$, $z=0$ essent. singularity,

$$(z+1)\cos\left(\frac{1}{z}\right) = (z+1)\left(1 - \frac{1}{2z^2} + \dots\right) =$$

$$= z + 1 - \frac{1}{2z} - \frac{1}{2z^2} + \dots$$

$$\operatorname*{Res}_{z=0}(f) = -\frac{1}{2},$$

(b) $$f = \frac{\sin^2\left(\frac{z}{2}\right)}{1-\cos z}, \quad 1-\cos z = 2\sin^2\left(\frac{z}{2}\right)$$

$\Rightarrow$ $f = 2$ has no singularities

(c) $$f = \frac{z+1}{(z-1)^3 z},$$

• $z = 0$, $$\operatorname{Res}(f) = \frac{1}{(-1)^3} = -1$$

• $z = 1$, $$f = \frac{1}{(z-1)^3}\left(1 + \frac{1}{z-1+1}\right) = \frac{1}{(z-1)^3}\cdot$$

$$\cdot\left(1 + 1 - (z-1) + (z-1)^2 + \dots\right) = \frac{2}{(z-1)^3} - \frac{1}{(z-1)^2} + \frac{1}{z-1}$$

+ regular part

$$\Rightarrow \quad \operatorname*{Res}_{z=1}(f) = 1.$$

(d) $$ze^{\frac{1}{z}} = z\left(1 + \frac{1}{z} + \frac{1}{2!\,z^2} + \cdots\right) =$$

$$= z + 1 + \frac{1}{2!\,z} + \frac{1}{3!\,z^2} + \cdots$$

$z = 0$, essent. singularity,

$$\operatorname*{Res}_{z=0}\left(ze^{\frac{1}{z}}\right) = \frac{1}{2!},$$

(e) $$(z+z^2)e^{\frac{1}{z}} = (z+z^2)\left(1 + \frac{1}{z} + \frac{1}{2!\,z^2} + \frac{1}{3!\,z^3} + \cdots\right) = z^2 + z + 1 + z + \frac{1}{2!} + \frac{1}{2!\,z} + \frac{1}{3!\,z} + O\left(\frac{1}{z^2}\right);$$

$\frac{1}{z^2}$ and higher

$z =$ essential singularity,

$$\operatorname*{Res}_{z=0}\left((z+z^2)e^{\frac{1}{z}}\right) = \frac{1}{2!} + \frac{1}{3!} = \frac{2}{3}$$

(f) $e^{z+\frac{1}{z}} = e^{z} e^{\frac{1}{z}} = \left(1+ z + \frac{z^2}{2!} + \frac{z^3}{3!} + \dots\right)$

$\left(1 + \frac{1}{z} + \frac{1}{2!\,z^2} + \frac{1}{3!\,z^3} + \dots\right) =$

$= \frac{1}{z}\left(1 + \frac{1}{2!} + \frac{1}{2!\,3!} + \dots + \frac{1}{n!\,(n+1)!} + \dots\right)$

+ other powers of z

$z=0$ – essential singularity,

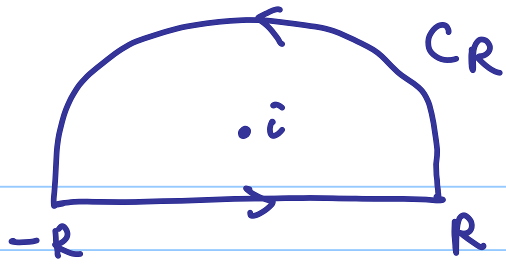
$\operatorname*{Res}_{z=0}\left(e^{z+\frac{1}{z}}\right) = \sum_{n=0}^{\infty} \frac{1}{n!\,(n+1)!}$,

4. (a) $\int_{-\infty}^{+\infty} \frac{\cos x}{x^2+1}\,dx$, solved during Monday's class

(b) $I = \int_{-\infty}^{+\infty} \frac{x\sin x}{(x^2+1)^2}\,dx = \frac{1}{2i} I_1 - \frac{1}{2i} I_2$

$I_1 = \int_{-\infty}^{+\infty} \frac{xe^{ix}}{(x^2+1)^2}\,dx$, $I_2 = \int_{-\infty}^{+\infty} \frac{xe^{-ix}}{(x^2+1)^2}\,dx$

I_1 :



$$\underset{\substack{\downarrow R\to\infty \\ I_1}}{\int_{-R}^{R}} + \underset{\substack{\downarrow R\to\infty \\ 0}}{\int_{C_R}} = 2\pi i \operatorname*{res}_{x=i}\left(\frac{xe^{ix}}{(x^2+1)^2}\right) = 2\pi i \operatorname*{res}_{x=i}\left(\frac{\frac{xe^{ix}}{(x+i)^2}}{(x-i)^2}\right)$$

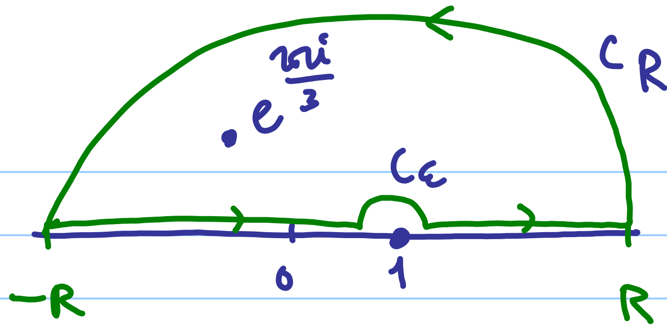
$$= 2\pi i\left(\frac{xe^{ix}}{(x+i)^2}\right)'\Bigg|_{x=i} = 2\pi i\left(\frac{i\cdot i\,e^{-1}}{(2i)^2} - \frac{2ie^{-1}}{(2i)^3} + \frac{e^{-1}}{(2i)^2}\right)$$

$$= 2\pi i\left(\frac{e^{-1}}{4} + \frac{e^{-1}}{4} - \frac{e^{-1}}{4}\right) = \frac{\pi e^{-1}}{2} i$$

$\Rightarrow \quad I_1 = \frac{\pi e^{-1}}{2} i$,

$I_2 = \overline{I_1}$, $\Rightarrow$ $\boxed{I = \frac{1}{2i} I_1 - \frac{1}{2i}\overline{I_1} = \frac{\pi e^{-1}}{2}}$

(c) $I = \text{p.v.}\int_{-\infty}^{+\infty} \frac{dx}{(x^3-1)} = \lim_{\substack{R\to\infty \\ \varepsilon\to 0}}\left(\int_{1+\varepsilon}^{R} + \int_{-R}^{1-\varepsilon}\right)$



$$\underbrace{\int_{-R}^{1-\varepsilon} + \int_{1+\varepsilon}^{R}}_{\to I} + \int_{C_\varepsilon} + \underbrace{\int_{C_R}}_{\substack{R\to\infty \\ \to 0}} = 2\pi i \operatorname*{res}_{z=e^{\frac{2\pi i}{3}}} \left(\frac{1}{z^3-1}\right)$$

$$\int_{C_\varepsilon} = \frac{1}{2} \oint_{|z-1|=\varepsilon} \frac{dz}{(z^3-1)} = -\pi i \operatorname*{res}_{z=1}(f)$$

$$\operatorname*{res}_{z=1} \frac{1}{(z^3-1)} = \left.\frac{1}{3z^2}\right|_{z=1} = \frac{1}{3}$$

$$\operatorname*{res}_{z=e^{\frac{2\pi i}{3}}} \left(\frac{1}{z^3-1}\right) = \frac{1}{3e^{\frac{4\pi i}{3}}} = \frac{1}{3} e^{\frac{2\pi i}{3}} =$$

$$= \frac{1}{3}\left(-\frac{1}{2} + i\frac{\sqrt{3}}{2}\right), \quad \Rightarrow$$

$$I - i\pi\frac{1}{3} = -i\frac{\pi}{3} - \frac{\pi}{\sqrt{3}}, \quad \boxed{I = -\frac{\pi}{\sqrt{3}}}$$

(d) $$\text{p.v.}\int_{-\infty}^{+\infty} \frac{\sin x}{x(x^2+1)}\,dx = \int_{-\infty}^{+\infty} \frac{\sin x}{x(x^2+1)}\,dx$$

5.

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6. (a) $$\frac{1}{z^2(z-1)^2} = P_0 + P_1$$

$$P_1 = \frac{1}{(z-1)^2} \left. \frac{1}{z^2} \right|_{z=1} + \frac{1}{z-1} \cdot \left. \left(\frac{1}{z^2}\right)' \right|_{z=1} =$$

$$= \frac{1}{(z-1)^2} - \frac{2}{z-1},$$

$$P_0 = \frac{1}{z^2} \left. \left(\frac{1}{z-1}\right)^2 \right|_{z=0} + \frac{1}{z} \left. \left(\left(\frac{1}{z-1}\right)^2\right)' \right|_{z=0} =$$

$$= \frac{1}{z^2} + \frac{2}{z}$$

$$\frac{1}{z^2(z-1)^2} = \frac{1}{(z-1)^2} - \frac{2}{z-1} + \frac{1}{z^2} + \frac{2}{z},$$

(b) $$\frac{z}{z^3(z-2)} = P_0 + P_2 = \frac{1}{z^4(z-2)}$$

$$P_0 = \frac{1}{z^2}\left(\frac{1}{z-2}\right)\Bigg|_{z=0} + \frac{1}{z}\left(\frac{1}{z-2}\right)'\Bigg|_{z=0}$$

$$= \frac{1}{z^2}\left(-\frac{1}{2}\right) + \frac{1}{4z}$$

$$P_2 = \frac{1}{4(z-2)},$$

$$\frac{z}{z^3(z-2)} = -\frac{1}{2z^2} + \frac{1}{4z} + \frac{1}{4(z-2)},$$