Math 185, Midterm 2

Section 1, 10-12am, N.Reshetikhin, July 30, 2009

Student's Name:

Student's i.d. number:

Problem	1	2	3	4	5	*Total*
Points	30	30	30	30	30	150
Grade						

1.*30 pnts* Find singularities of

$$f(z) = \begin{cases} \frac{\sin(z-2)\sin(\frac{1}{z})}{(z-2)(z-1)^2}, & z \neq 2 \\ 2, & z = 2 \end{cases}$$

and describe their types.

Use Riemann principle :

$z = 2$, removable

$z = 1$, pole of order 1

$z = 0$, essential

2. *30 pnts* Find Laurent series about $z = 1$ for

$$\frac{1}{(z^2-1)(z-2)}$$

$$\frac{1}{(z-1)(z+1)(z-2)} = -\frac{1}{2(z-1)} + \frac{1}{6(z+1)} + \frac{1}{3(z-2)}$$

(using residues)

$$\frac{1}{z+1} = \frac{1}{2+(z-1)} = \frac{1}{2}\frac{1}{1+\frac{z-1}{2}} = \sum_{n=0}^{\infty} \frac{(-1)^n}{2^{n+1}}(z-1)^n,$$

$$\frac{1}{z-2} = \frac{1}{(z-1)-1} = -\frac{1}{1-(z-1)} = -\sum_{n=0}^{\infty}(z-1)^n,$$

$$\frac{1}{(z-1)(z+1)(z-2)} = -\frac{1}{2(z-1)} + \sum_{n=0}^{\infty}\left(\frac{(-1)^n}{6\cdot 2^{n+1}} - \frac{1}{3}\right)(z-1)^n$$

3. *30 pnts* Let $f(x,y) = u(x,y) + iv(x,y)$ be an analytic function in a domain D. Can $v(x,y)$ achieve a maximum inside D?

Consider $g = e^{-if}$,

$|g| = e^{v}$, f is analytic $\Rightarrow$ g is analytic.

By max. mod. thm $|g|$ can not have a maximum inside D.

Because e^{v} is mononic function, v also can not have a max inside D.

4. *30 pnts* Find isolated singular points of

$$f(z) = \frac{1}{1 - \cos(2z)}$$

describe their type and find residues.

- $z = \pi n$, solutions to $\cos(2z) = 1$
- poles of order 2
- $f(z+\pi) = f(z) \Rightarrow$ all residues are the same

Find the residue at $z = 0$

$$f(z) = \frac{1}{1-\cos 2z} = \frac{1}{2\sin^2 z}$$ is

an even function of $z \Rightarrow$

$$\operatorname*{Res}_{z=0} f(z) = 0 \Rightarrow \operatorname*{Res}_{z=\pi n} f(z) = 0$$

5. *30 pnts* For real a compute

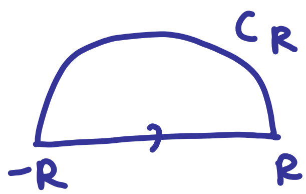
$$\int_{-\infty}^{\infty} \frac{\cos(x)}{(x^2+a^2)^2}dx =$$

$$= \int_{-\infty}^{+\infty} \frac{e^{ix}}{(x^2+a^2)^2}dx, \quad \exists a>0 \ \left((-a)^2=a^2\right)$$

because $\sin x$ is an odd function.

Converges by comp. test with $\frac{1}{x^4}$.

One pole of order 2 in the upper half-plane at $x = ia$



$$(*) \quad \int_{-R}^{R} + \int_{C_R} = 2\pi i \operatorname*{Res}_{x=ia} \frac{e^{ix}}{(x+ia)^2(x-ia)^2}$$

$\int_{C_R} \to 0$ by the prop $\left(e^{iz} \text{ is bounded on } C_R\right)$

$$\operatorname*{Res}_{x=ia} \frac{e^{ix}}{(x+ia)^2(x-ia)^2} = \left(\frac{e^{ix}}{(x+ia)^2}\right)'\Bigg|_{x=ia} = -i\frac{1}{4}\left(\frac{1}{a^2}+\frac{1}{a^3}\right)e^{-a}$$

$\Rightarrow$ taking $R \to \infty$ in $(*)$ we obtain

$$\int_{-\infty}^{+\infty} \frac{e^{ix}}{(x^2+a^2)^2}dx = \frac{\pi}{2}\left(\frac{1}{a^2}+\frac{1}{a^3}\right)e^{-a}$$