Math 185, Midterm 1

Section 1, 10-12am, N.Reshetikhin, July 9, 2009

Student's Name:

Student's i.d. number:

Problem	1	2	3	4	5	*Total*
Points	30	30	30	30	30	150
Grade						

1.*30 pnts* Is the function $f(z) = Im \cos z$ analytic? Justify your answer.

1 solution: $f = u \quad (v = 0)$,

it is analytic only if $u_x = v_y$, $u_y = -v_x$

i.e. $u_x = u_y = 0$, i.e. $u =$ const.

(only constants are real valued analyt. fncns) $\operatorname{Im} \cos z \neq$ const $\Rightarrow$ it is not

2 solution $\cos(x+iy) = \cos x \cos iy - \sin x \sin(iy)$

$= \cos x \operatorname{ch} y - i \sin x \operatorname{sh} y$,

$f = \operatorname{Im} \cos z = \sin x \operatorname{sh} y$, $u = \sin x \operatorname{sh} y$, $v = 0$

$u_x = \cos x \operatorname{sh} y \neq 0 \quad (\Rightarrow u_x \neq v_y)$

f is not analytic

2.*30 pnts* Does the function $f(z) = Im(\ln(z))$ defined on the upper half plane have the limit as $z \to 0$ from the upper-half plane?

$z = \varepsilon e^{i\theta}$, $\quad 0 < \theta < \pi$

Check if $L_\theta = \lim_{\varepsilon \to +0} f(\varepsilon e^{i\theta})$ exists

$f(\varepsilon e^{i\theta}) = \operatorname{Im}(\ln z) = \operatorname{Im}(\ln \varepsilon + i\theta) = \theta$

$L_\theta = \theta$ exists for all $0 < \theta < \pi$

but depends on $\theta \Rightarrow$

$\lim_{z \to 0} \operatorname{Im} \ln z$ does not exist

3. The region of convergence : unit disk with the boundary and $z=1$ is removed:

3. *30 pnts* Find the region of convergence of the series

$$\sum_{n=1}^{\infty} \sin\left(\pi \frac{n}{n+1}\right) z^n$$

$$\sin\left(\pi \frac{n}{n+1}\right) = \sin\left(\pi\left(1 - \frac{1}{n+1}\right)\right) = \sin\left(\frac{\pi}{n+1}\right)$$

1. The Ratio Test:

$$\lim_{n\to\infty} \left|\frac{c_{n+1}}{c_n}\right| = \lim_{n\to\infty} \frac{\sin\left(\frac{\pi}{n+2}\right)}{\sin\left(\frac{\pi}{n+1}\right)} \overset{\text{l'Hospital}}{=} \lim_{n\to\infty} \frac{\cos\left(\frac{\pi}{n+2}\right)}{\cos\left(\frac{\pi}{n+1}\right)} = 1$$

$$\Rightarrow \boxed{R=1}$$

2. $|z|=1$

$a_n = z^n$, $b_n = \sin\left(\frac{\pi}{n+1}\right) > 0$, monotonically decreasing

$\sum_{n=1}^{N} a_n = \sum_{n=1}^{N} z^n = z \frac{1-z^N}{1-z}$ bounded when $|z|=1$ but $z=1$

$\Rightarrow$ the series converges conditionally for all $z \neq 1$. when $z=1$ it diverges by the integral test (above)

4. *30 pnts* The function $f(z)$ is entire. Is the function

$$g(z) = \begin{cases} \frac{f(z)-f(0)}{z} & , \quad z \neq 0 \\ f'(0) & , \quad z = 0 \end{cases}$$

also entire?

1. Analytic for all $z \neq 0$ because it is a sum of products of fncns which are analytic when $z \neq 0$

2. f is analytic at $z=0$ $\Rightarrow$

$f(z) = \sum_{n=0}^{\infty} \frac{f^{(n)}(0)}{n!} z^n$ in a nbd. of z

$\Rightarrow$ $g(z) = \sum_{n=1}^{\infty} \frac{f^{(n)}(0)}{n!} z^{n-1}$ (converges, i.e. partial sums have the limit because the Taylor series for f converges)

$\Rightarrow$ $g(z)$ is analytic (satisfies the Cauchy-Rieman equations) at $z=0$.

5. *30 pnts* Compute the integral

$$\int_C \sin^2(x+iy)dz$$

where C is:

$\sin^2 z$ is an entire function.

(we know it, but one can always check that Cauchy - Riemann equations hold)

$\sin^2 z = \frac{1}{2}(1-\cos 2z)$,

an antiderivative for $\sin^2 z$:

$F(z) =$ "$\int \sin^2 z \, dz$" $= \frac{z}{2} - \frac{1}{4}\sin 2z$,

$\Rightarrow$

$$\int_C \sin^2(x+iy)\,dz = \int_0^{Re^{i\theta}} \sin^2 z\,dz = F(Re^{i\theta}) - F(0)$$

$$= \frac{Re^{i\theta}}{2} - \frac{1}{4}\sin\left(2Re^{i\theta}\right),$$