Relative Rounding

Chikara Nakayama and Arthur Ogus

September 24, 2009

Contents

Introduction

Our goal in this note is to study singularities of mappings of toric varieties, and more generally, logarithmic analytic spaces. We shall show that the introduction of polar coordinates—which effectuates a kind of real blowing up—smoothes out a wide class of such mappings, rendering them locally trivial (submersive) in the topological category. Suitably globalized, this technique provides a powerful tool for analyzing the geometry of degenerations. Some cohomological manifestations of this technique have already been studied in [8], with applications to monodromy and vanishing cycles in [6] and [13].

To state our main local result, we let $(\mathbf{R}_{\geq}, \cdot)$ denote the multiplicative monoid of nonnegative real numbers, endowed with its usual topology. Let

Q be a monoid and let X_Q be the set of monoid homomorphisms from Q to $(\mathbf{R}_{\geq}, \cdot)$. Each $q \in Q$ defines a function

$$e_q \colon X_Q \to \mathbf{R}_{\geq} \colon x \mapsto x(q),$$

and we endow X_Q with the weak topology and monoid structure defined by the set of such functions.

A monoid Q is said to be *fine* if it is commutative, finitely generated and cancellative, and to be *sharp* if its group of units Q^* is trivial. The following result is a simple but important special case of our main theorem.

Proposition 0.1 *Let Q be a fine sharp monoid and let p be a nonzero element of Q. Then the map $e_p \colon X_Q \to \mathbf{R}_{\geq}$ is homeomorphic to a product map. That is, there exist a topological space Z and a commutative diagram*

$$\begin{tikzcd}
X_Q \arrow[r] \arrow[dr, "e_p"'] & Z \times \mathbf{R}_{\geq} \arrow[d, "pr_2"] \\
 & \mathbf{R}_{\geq}
\end{tikzcd}$$

in which the horizontal map $X_Q \to Z \times \mathbf{R}_{\geq}$ is a homeomorphism.

Somewhat more generally, let $V_Q := \mathbf{R} \otimes Q^{gp}$ and let C_Q be the real subcone of V_Q spanned by Q (that is, the set of linear combinations of elements of Q with nonnegative coefficients). Then each $c \in C_Q$ also defines a function $e_c \colon X_Q \to \mathbf{R}_{\geq}$, and in fact Proposition 0.1 is true for all nonzero $c \in C_Q$.

It is natural to ask if Proposition 0.1 holds more generally for a suitable class of morphisms $\theta \colon P \to Q$ of finitely generated monoids or cones. Some hypotheses are clearly required; for example, if $\theta \colon \mathbf{N}^2 \to \mathbf{N}^2$ is the map sending (a, b) to $(a, a + b)$, the dimension of the fibers of X_θ is not constant, so X_θ cannot be homeomorphic to a projection mapping. Our generalization of Proposition 0.1 depends on K. Kato's important notion of *exactness*. After a review of this notion and some of its variants (see Definition 2.1), we shall prove the following result.

Theorem 0.2 *Let P and Q be fine monoids and let $\theta \colon C_P \to C_Q$ be an injective, exact, and locally exact morphism of their corresponding real cones.*

Then there exist a topological space Z and a commutative diagram:

$$\begin{tikzcd}
X_Q \arrow[r] \arrow[dr, "X_\theta"'] & Z \times X_P \arrow[d, "pr_2"] \\
& X_P
\end{tikzcd}$$

in which the horizontal map is a homeomorphism.

Our study of these questions, raised in [8, B.3] under the rubric "relative rounding," was motivated by logarithmic geometry. In particular, Theorem 0.2 and some standard arguments, which we shall review in §4, imply the following.

Theorem 0.3 *Let $f: X \to Y$ be a smooth and exact morphism of fine log analytic spaces. Then the associated map of Betti realizations $f_{log}: X_{log} \to Y_{log}$ is a topological submersion. That is, locally on X_{log} and Y_{log} it is homeomorphic to a projection from a product.*

It follows (see Theorem 5.1) that if f is exact, smooth, separated, and proper, then f_{log} is a "fiber bundle": locally on Y_{log}, $f_{log}: X_{log} \to Y_{log}$ is a projection from a product. This result can be used to give a new and more direct proof of some of the main theorems of [8]—for example, that the cohomology sheaves $R^q f_{log*}(\mathbf{Z})$ are locally constant on Y_{log}. As we shall see in Theorem 3.7, many of these results apply even more generally, to some log structures which are not "coherent," but only "relatively coherent." Such log structures arise naturally in the study of some degenerations of Calabi Yau varieties and have been considered already in [13] and [3].

A key role in our proof is played by the *moment map*, which gives a linear description of the topological space X_Q. Let S be any finite set of generators for a fine monoid Q. Then the associated moment map is defined by:

$$\mu_S: X_Q \to C_Q : x \mapsto \sum_{s \in S} x(s)s.$$

It is proved in [1, §4.2] that this map is a homeomorphism, compatible with the faces of X_Q and C_Q (see Theorem 1.4 for a more precise and general statement). The difficulty in applying this construction is that it is not functorial with respect to morphisms of monoids. However our methods show that a certain functoriality can be "forced," as explained in Proposition 2.11.

Our paper is organized as follows. Section 1 is a review of some basic facts about cones and monoids, including a new treatment of the moment mapping which will play a crucial role. The heart of our paper is Section 2, which contains the proof of Theorem 0.2. Section 3 contains a brief introduction to log geometry and the proof of Theorem 0.3. Section 4 describes a generalization of our main results to "idealized" monoids and log analytic spaces, which can be useful in dealing with the strata that arise naturally from log structures. The last section is devoted to applications to cohomology, including a discussion of monodromy, orientation and duality. There is also an appendix with an elementary proof that higher direct images of a locally constant sheaf by a separated proper submersion are again locally constant.

Many special cases and consequences of our results have been known for a long time. The case of semistable reduction for example, has a long history, going back at least to [4, Exposé I] (before the invention of log structures) and more recently was treated in the context of log geometry by S. Usui [18], [17]. Cohomological and homotopical versions appear in [8], [14], and [13].

We are grateful to many mathematicians who helped and encouraged us in this work. Particular thanks go to Robion Kirby and Martin Olsson, for many discussions, to Bernd Sturmfels for pointing us to [15], and to Luc Illusie for his help and advice with Section 5.

1 The moment map

We begin by reviewing some well-known facts about the structures of X_Q and C_Q which will be important in the proofs and applications of our main results. All monoids discussed in this paper will be commutative, and a monoid is said to be *integral* if satisfies the cancellation law.

Recall that an *ideal* in a monoid Q is a subset J which is invariant under translation by elements of Q, and that an ideal is *prime* if its complement is a submonoid of Q. The submonoids F of Q whose complements are prime ideals are the *faces* of Q. If F is a face of Q, then $q_1 + q_2$ belongs to F if and only if q_1 and q_2 belong to F. For $q \in Q$, we denote by $\langle q \rangle$ the smallest face of Q containing q, *i.e.*, the set of all f such that there exist $f' \in Q$ and $n \in \mathbf{N}$ such that $f + f' = nq$. If $h\colon Q \to Q'$ is a homomorphism of monoids and F'

is a face of Q', then $h^{-1}(F')$ is a face of Q. If S is a subset of Q, we denote by $\lambda_S\colon Q \to Q_S$ the *localization of* Q *by* S, *i.e.*, the universal map from Q to a monoid in which the elements of S become invertible. If F is a face of Q, then $F = \lambda_F^{-1}(Q_F^*)$. A basic fact from duality theory for monoids asserts that if F is a face of a *fine* monoid Q, there is a homomorphism $h\colon Q \to (\mathbf{N}, +)$ such that $F = h^{-1}(0)$ [12].

Remark 1.1 Let Q be an integral monoid, let $V_Q := \mathbf{R} \otimes Q^{gp}$, and let $C_Q \subseteq V_Q$ be the real cone spanned by Q, *i.e.*, the set of linear combinations of elements of Q with nonnegative coefficients. Then the map $Q \to C_Q$ induces a bijection from the set of faces of C_Q to the set of faces of Q. To check this, first note that we may assume without loss of generality that Q^{gp} is torsion free, since the map from Q to its image Q' in Q^{gp}/Q^{gp}_{tor} induces a bijection $\operatorname{Spec} Q' \to \operatorname{Spec} Q$ and an isomorphism $C_Q \to C_{Q'}$. When Q^{gp} is torsion free, the map $Q \to C_Q$ is injective, and we write it as an inclusion. Note that if G is a face of C_Q, then necessarily G is invariant under the action of $\mathbf{R}_\geq$. Indeed, if $r \in \mathbf{R}_\geq$ and $g \in G$, choose $n \in \mathbf{N}$ with $n \geq r$. Then $ng \in G$, and $ng = rg + (n-r)g$, hence rg belongs to G. Now any $g \in G$ can be written $g = \sum r_q q$ with $r_q \in \mathbf{R}_\geq$ and $q \in Q$, and it follows that $q \in G$ whenever $r_q > 0$. Hence G is the cone spanned by $G \cap Q$, a face of Q. Conversely, we claim that if F is any face of Q, then C_F is a face of C_Q and $C_F \cap Q = F$. Suppose that $c := \sum r_f f$ and $c' := \sum r_{f'} f'$ belong to C_Q and $c + c' \in C_F$. There is a fine submonoid Q' of Q such that $C_{Q'}$ contains all f and f' with r_f or $r_{f'}$ nonzero. Choose a homomorphism $h\colon Q' \to \mathbf{N}$ such that $h^{-1}(0) = F \cap Q'$. Then h induces a homomorphism $C_h\colon C_{Q'} \to (\mathbf{R}_\geq, +)$, and one sees immediately that $C_h^{-1}(0) = C_{F\cap Q'}$. It follows that $C_{F \cap Q'}$ is a face of $C_{Q'}$ and hence that c and c' belong to $C_{F\cap Q'} \subseteq C_F$. Hence C_F is a face of C_Q. Similarly, if $q \in Q \cap C_F$, say $q = \sum r_f f$ with $f \in F$, there is a fine submonoid Q' of Q containing q and all f with $r_f > 0$. Then choosing h as above, we see that $h(q) = C_h(q) = 0$, and hence $q \in F$.

An ideal J of C_Q is said to be a *radical ideal* if $rq \in J$ whenever $q \in J$ and $r > 0$. Then if $c \in C_Q \setminus J$, it follows that $\langle c \rangle \subseteq C_Q \setminus J$. Hence $C_Q \setminus J$ is the union of the faces F of C_Q not meeting J, and J is the intersection of the prime ideals which contain J.

If Σ is a subset of Q, let $Z(\Sigma)$ be the set of $x \in X_Q$ which vanish on Σ. Then $Z(\Sigma)$ is closed ideal in the topological monoid X_Q, and $Z(\Sigma) = Z(I)$, where I is the ideal of Q generated by Σ. In fact the set of all subsets of the form $Z(I)$ defines another topology on X_Q, the *Zariski topology.* The irreducible closed sets for this topology are those defined by the prime ideals $\mathfrak{p}$ of Q. The complement F of a prime ideal $\mathfrak{p}$ of Q (resp. C_Q) is a face of Q (resp. C_Q) and there is a natural embedding $i_F \colon X_F \to X_Q$, where

$$i_F(x)(q) := \begin{cases} x(q) & \text{if } q \in F \\ 0 & \text{otherwise.} \end{cases}$$

The image of i_F is precisely $Z(\mathfrak{p})$ and we will allow ourselves to identify X_F with $Z(\mathfrak{p}) \subseteq X_Q$. Then we have identifications

$$X_F^* := X_{F^{gp}} = \{x \in X_Q : x^{-1}(\mathbf{R}^*_{\geq}) = F\},$$

and X_Q is the disjoint union $X_Q = \cup X_F^*$ as F ranges over the faces of Q. For each $x \in X_Q$, let $F(x) := \{q \in Q : x(q) \neq 0\}$, the face of Q such that $x \in X^*_{F(x)}$.

Observe that if S is any finite set of generators of C_Q, the map $X_Q \to \mathbf{R}^S$ sending x to the sequence $(x(s) : s \in S)$ is injective and that its image is a closed subset of $\mathbf{R}^S$; furthermore X_Q has the topology and monoid structure induced from $\mathbf{R}^S$.

The subset

$$X_Q^* := \operatorname{Hom}(Q, \mathbf{R}^*_{\geq}) = \operatorname{Hom}(Q^{gp}, \mathbf{R}_{\geq})$$

is a submonoid of X_Q and in fact is a topological group, isomorphic to a product of copies of $(\mathbf{R}^*_{\geq}, \cdot)$ (which in turn is isomorphic to the topological group $(\mathbf{R}, +)$ via the logarithm map). It acts naturally on X_Q, and each X_F^* is stable under this action. In fact each X_F^* is also naturally a Lie group. If $f \in F$ and $x \in X_Q$, $e_f(x) > 0$ if $x \in X_F^*$. Then $\log(e_f)$ is a well-defined function on X_F^*, and its differential $d\log(e_f)$ is an invariant differential form. Then $f \mapsto d\log(e_f)$ induces a natural isomorphism from $V_F := \mathbf{R} \otimes F^{gp}$ to the space of invariant differential forms on X_F^*, and hence an isomorphism from the Lie algebra of X_F^* to $V_F^\vee := \operatorname{Hom}(F^{gp}, \mathbf{R})$. To simplify the notation we write these isomorphisms as identifications. Thus if $f \in F$, we view $1 \otimes f \in V_F$ as an invariant differential form on X_F^*, and if $\phi \in V_F^\vee$ we view ϕ as an invariant vector field on X_F^*. With this notation, we have the formula:

$$de_f = e_f \otimes f. \tag{1}$$

Similarly, the interior C_F^o of the cone spanned by F has a natural structure of a C^∞ manifold, induced from the inclusion $C_F^o \subseteq V_F$, and the invariant vector fields on the ambient space V_F are naturally identified with elements of V_F.

Let $\mathbf{R}[Q]$ (resp. $\mathbf{R}[C_Q]$) be the real monoid algebra of Q (resp. C_Q). Its underlying vector space is just the set of real linear combinations of elements of Q (resp. C_Q), which we also refer to as *cycles* in Q (resp. C_Q). We say a cycle is *effective* if its coefficients are all nonnegative; the set of effective cycles is a submonoid of $\mathbf{R}[Q]$ (resp. $\mathbf{R}[C_Q]$) under addition (and multiplication). Each cycle $A := \sum a_q q$ of $\mathbf{R}[C_Q]$ defines a function $e_A\colon X_Q \to \mathbf{R}_{\geq}$, where $e_A(x) = \sum a_q e_q(x) : q \in C_Q$.

Definition 1.2 *The moment map of an effective cycle A in C_Q is the function $X_Q \to C_Q$ defined by*

$$\mu_A(x) := \sum_{q \in C_Q} a_q x(q) q.$$

A morphism $\theta\colon C_P \to C_Q$ of finitely generated cones induces a morphism

$$X_\theta\colon X_Q \to X_P : x \mapsto x \circ \theta.$$

Then X_θ is continuous with respect to the standard topology and the Zariski topology. A cycle A in C_P defines a cycle $\theta_*(A)$ in C_Q, where

$$\theta_*(A)_q := \sum_{p:\theta(p)=q} a_p,$$

and there are commutative diagrams:

$$\begin{tikzcd}
X_Q \arrow[d, "X_\theta"'] \arrow[dr, "e_{\theta_*(A)}"] & \\
X_P \arrow[r, "e_A"] & \mathbf{R}_{\geq}
\end{tikzcd}
\qquad
\begin{tikzcd}
X_Q \arrow[r, "\mu_{\theta_*(A)}"] \arrow[d] & C_Q \\
X_P \arrow[r, "\mu_A"] & C_P. \arrow[u]
\end{tikzcd}$$

An effective cycle A in C_Q can be viewed as a function $C_Q \to \mathbf{R}_{\geq}$ with finite support. Thus if F is a face of Q, the restriction $A_{|F}$ of A to C_F can be viewed as an effective cycle in C_F. If $x \in X_F \subseteq X_Q$, then $e_q(x) = 0$ if $q \notin F$, so $e_A \circ i_F = e_{A_{|F}}$. Furthermore,

$$\mu_A(x) = \sum_{q \in C_Q} a_q x(q) q = \sum_{f \in C_F} a_f x(f),$$

and the following diagram commutes.

$$\begin{tikzcd}
X_F \arrow[r, "i_F"] \arrow[dr, "\mathrm{id}"'] & X_Q \arrow[r, "\mu_A"] & C_Q \\
& X_F \arrow[r, "\mu_{A_{|F}}"] & C_F \arrow[u]
\end{tikzcd} \tag{2}$$

This will allow us to identify $e_{A_{|F}}$ with $e_{A|_{X_F}}$ and $\mu_{A_{|F}}$ with $\mu_{A|_{X_F}}$. Note that if the support S of A generates C_Q, then $S \cap C_F$ necessarily generates C_F, and if $x \in X_F$ (resp. X_F^*), then $\mu_A(x)$ belongs to C_F (resp. C_F^o).

The following result describes the differential properties of the moment map.

Proposition 1.3 *Let A be an effective cycle in C_Q, let $S \subseteq C_Q$ be its support, and let F be a face of Q.*

1. *The restriction of the moment map μ_A to X_F^* is the differential of the restriction of function e_A to X_F^*.*
2. *Let x be a point of X_F^* and consider the derivative of μ_A at x:*

$$\tau_x := T_x(\mu_A)\colon T_x(X_F^*) \to T_{\mu_A(x)}(C_F^o) \quad \textit{i.e.,} \quad V_F^\vee \to V_F.$$

Then the associated bilinear form β_x on $V_F^\vee$:

$$\beta_x(\phi, \psi) := \psi(\tau_x(\phi))$$

is symmetric and positive semi-definite. If $S \cap C_F$ generates C_F, then β_x is positive definite and τ_x is an isomorphism.

Proof: As we have seen,

$$e_A \circ i_F = \sum_{f \in F \cap S} a_f e_f,$$

so by the formula (1) for de_f on X_F^*,

$$de_{A_{|F}} = \sum_{f \in F \cap S} a_f de_f = \sum_{f \in F \cap S} a_f e_f \otimes f = \mu_{A_{|F}}.$$

Then

$$T_x(\mu_A) = \sum a_f de_f \otimes f = \sum a_f e_f \otimes f \otimes f \in \mathbf{R}[C_F] \otimes \mathrm{Hom}(V_F^\vee, V_F).$$

In particular, for $x \in X_F^*$ and $\phi \in V_F^\vee$,

$$\tau_x(\phi) = \sum_f a_f x(f)\phi(f) f,$$

and

$$\beta_x(\phi, \psi) = \sum_f a_f \phi(f) x(f) \psi(f).$$

Thus β_x is symmetric. Since a_f and $x(f)$ are nonnegative, it is also positive semi-definite. If $S \cap C_F$ generates C_F, then $S \cap C_F$ spans V_F, so β_x is positive definite and consequently τ_x is an isomorphism. $\square$

The following result is well-known. We present a proof, based on a combination of ideas from [1], [8, A.1.1], and [15] (see the discussion of Birch's Theorem, 1.10) in a form that will be useful for us later.

Theorem 1.4 *Let Q be a fine monoid and A an effective cycle in C_Q whose support generates C_Q. The moment map μ_A is a homeomorphism*

$$\mu_A \colon X_Q \to C_Q$$

compatible with the stratifications of X_Q and C_Q by faces.

Proof: We have already observed that if F is a face of Q, μ_A maps X_F to C_F and X_F^* to C_F^o, so that μ_A is compatible with the stratifications by faces. Thus to prove that μ_A is injective it will suffice to prove that for each face F, μ_A induces an injection $X_F^* \to C_F^o$. To simplify notation, we may and shall assume that $F = Q$. We have an isomorphism

$$\exp_Q \colon V_Q^\vee \to X_Q^* \quad : \quad \phi \mapsto \exp \circ \phi.$$

If ϕ and ϕ' are distinct points of $V_Q^\vee$, let $\psi := \phi' - \phi$, and for each real number t, let $\phi_t := \phi + t\psi$ and $x_t = \exp_Q(\phi_t)$. Then it follows from (2) of Proposition 1.3 that the derivative of $\psi(\mu_A(x_t))$ with respect to t is $\beta_{x_t}(\psi, \psi) > 0$. Explicitly,

$$\psi(\mu_A(x_t)) = \sum_s a_s \psi(s) \exp\left(\phi(s) + t\psi(s)\right),$$

and the derivative with respect to t is

$$\sum_s a_s(\psi(s))^2 x_t(s) > 0.$$

Thus $\psi(\mu_A(x_t))$ is an increasing and hence injective function of t. This implies that $\mu_A(\exp\circ\phi) \neq \mu_A(\exp\circ\phi')$.

Next let us prove that μ_A induces a surjection from X^*_Q to the interior of C_Q. The key point is the following lemma.

Lemma 1.5 *Let c be an interior point of C_Q and let*

$$\Psi_c := e_c \exp(-e_A) \colon X_Q \to \mathbf{R}_\geq.$$

Then for every positive number r, the set $X_{A,c}(r)$ of all $x \in X_Q$ such that $\Psi_c(x) \geq r$ is compact and contained in X^*_Q.

Proof: Let S be the support of A and write $c = \sum\{c_s s : s \in S\}$ with each $c_s > 0$. If F is a proper face of Q and $x \in X_F$, then $x(s) = 0$ for some $s \in S$, hence $e_c(x) = \prod x(s)^{c_s} = 0$. Thus Ψ_c vanishes on $X_Q \setminus X^*_Q$. It follows that $X_{A,c}(r) \subseteq X^*_Q$ when $r > 0$.

Since $X_{A,c}(r)$ is a closed subset of a Euclidean space, its compactness will follow if we prove it is bounded. Since $X_{A,c}(r)$ is contained in X^*_Q, we can let $\psi := -\log\Psi_c$, and it suffices to prove that for any real number m, the set $X^*_{A,c}(m)$ of points of X^*_Q where $\psi < m$ is bounded. For $x \in X^*_Q$,

$$\psi(x) = e_A(x) - \log e_c(x) = \sum_s \Big(a_s x(s) - c_s \log x(s)\Big) = \sum \psi_s(x(s)),$$

where $\psi_s(t) := a_s t - c_s \log t$. Since a_s and c_s are both nonnegative, the function ψ_s is bounded below, and we can find a positive number m' such that $\psi_s(t) \geq -m'$ for all t and all s. If $x \in X^*_{A,c}(m)$ we have, for each $s \in S$,

$$\psi_s(x(s)) = \psi(x) - \sum_{s' \neq s} \psi_{s'}(x(s')) \leq \psi(x) + |S|m' \leq m'' := m + |S|m'.$$

On the other hand, the restriction of ψ_s to $[c_s/a_s, \infty)$ is an increasing function which tends to ∞. Hence for each s, there is a constant m_s such that $t < m_s$ whenever $\psi_s(t) < m''$. Then if $x \in X^*_{A,c}(m)$, each $x(s) < m_s$. This shows that $X^*_{A,c}(m)$ is indeed bounded.

□

Now choose a point x_0 in X_Q^*. Then $r := \Psi_c(x_0) > 0$, and the set $X_{A,c}(r/2)$ is nonempty and compact. It follows that Ψ_c has a maximum at some point x of $X_{A,c}(r/2)$. But then in fact $\Psi_c(x)$ is the maximum of Ψ_c on all of X_Q^*. Thus Ψ_c has a critical point at x, and hence so does $\log \Psi_c$. Then $d \log \Psi_c(x) = 0$. By Proposition 1.3,

$$d \log \Psi_c = d \log e_c - de_A = c - \mu_A,$$

so $\mu_A(x) = c$.

It is clear from the definition that μ_A is continuous, and we have now proved that it is a bijection. It remains to prove that its inverse is also continuous. We shall make use of the following convenient elementary lemma.

Lemma 1.6 *Let $f\colon X \to Y$ be a continuous map of Hausdorff spaces.*

1. *Suppose that there is a chain $X_1 \subseteq X_2 \subseteq \cdots \subseteq X$ of compact subsets of X which covers X and whose image in Y satisfies the ascending chain condition locally on Y. Then f is a homeomorphism if it is bijective.*

2. *In particular, suppose that X and Y are closed subsets of some Euclidean space $\mathbf{R}^n$. Then f is a homeomorphism if and only if it is bijective and f^{-1} takes bounded sets to bounded sets.*

3. *If X and Y are closed subsets of some Euclidean space, then every continuous map from X to Y takes bounded sets to bounded sets.*

Proof: To prove (1), let X'_n be the image of X_n in Y. The hypothesis says that Y admits a covering by open sets V each of which is contained in some X'_n. Now suppose that Z is a closed subset of X. Then $Z \cap X_n$ is compact, and hence its image $f(Z \cap X_n) = f(Z) \cap X'_n$ is compact, hence closed in Y. For each V in the aforementioned covering, $f(Z) \cap V = f(Z) \cap X'_n \cap V$ for some n, and hence $f(Z) \cap V$ is closed in V. Since this is true for every V, $f(Z)$ is closed in Y, and hence f is a closed mapping and f^{-1} is continuous.

For (2), suppose f^{-1} takes bounded sets to bounded sets. Let X_n be the closed ball of radius n about some $x_0 \in X$, let X'_n be the image of X_n in Y, and let V be any (finite) open ball in Y. Then by hypothesis $f^{-1}(V)$ is bounded in X, and hence contained in X_n for some n. Then $V \subseteq X'_n$, and the chain $X_1 \subseteq X_2 \cdots$ satisfies the hypothesis of (1). The converse follows from (3).

For (3), suppose that $f\colon X \to Y$ is continuous and that B is a bounded subset of X. Then its closure is closed and bounded, hence compact, hence its image is compact, hence bounded. Hence the image of B is also bounded. $\square$

Since μ_A is a continuous bijective map between closed subsets of Euclidean spaces, the following lemma will show that it is a homeomorphism.

Lemma 1.7 *If the support S of A generates C_Q, then μ_A^{-1} takes bounded subsets of C_Q to bounded subsets of X_Q.*

Proof: Let $h\colon C_Q \to \mathbf{R}_{\geq}$ be a local homomorphism so that $h(q) = 0$ if and only if $q \in C_Q^*$. Let $S^* := S \cap C_Q^*$ and let $S^+ := S \setminus S^*$. Then there exists a positive number r such that $h(s) \geq r$ for each $s \in S^+$, and we may assume that $r = 1$. Furthermore, if B is a bounded subset of C_Q, $h(B)$ is bounded, say by b. Let $a := \min\{a_s : s \in S\}$. If $\mu_A(x) \in B$,

$$b \geq h(\mu_A(x)) = \sum_{s \in S} a_s x(s) h(s) \geq \sum_{s \in S^+} a_s x(s)$$

It follows that $x(s) \leq b/a$ whenever $s \in S^+$. Consider the map $x'\colon Q \to \mathbf{R}_{\geq}$ defined by $x'(q) = x(q)$ if $q \in C_Q^*$ and $= 0$ otherwise. Then $x' \in X_Q$, and

$$\mu_A(x) = \sum_{s \in S^*} a_s x(s) s + \sum_{s \in S^+} a_s x(s) s = \mu_A(x') + \sum_{s \in S^+} a_s x(s) s.$$

Since $x(s) \leq b/a$ for $s \in S^+$, it follows that there is a bounded set B' of C_Q which contains $\sum\{a_s x(s) s : s \in S^+\}$ whenever $\mu_A(x) \in B$. It follows that there is also a bounded set B'' of C_Q which contains $\mu_A(x')$ whenever $\mu_A(x) \in B$. Thus it suffices to show that x' is bounded with $\mu_A(x')$, and we are reduced to the case $Q = Q^*$. In this case $C_Q = V_Q$, and we have again the isomorphism $\exp\colon V_Q^\vee \to X_Q^*$. Proposition 1.3 shows that the differential of the map $\mu_A\colon X_Q^* \to C_Q^o$ is an isomorphism, so by the inverse function theorem μ_A is an open mapping, hence a homeomorphism. Then (3) of Lemma 1.6 implies that μ_A^{-1} takes bounded sets to bounded sets. $\square$

$\square$

2 Relative rounding

This section is devoted to a precise formulation and proof of Theorem 0.2. We begin by reviewing the notion and properties of exactness. The main ideas are taken from the appendix of [6].

Definition 2.1 *A morphism $\theta \colon P \to Q$ of integral monoids is:*

1. *local if $\theta^{-1}(Q^*) = P^*$,*
2. *exact if the diagram*

$$\begin{tikzcd} P \arrow[r, "\theta"] \arrow[d] & Q \arrow[d] \\ P^{gp} \arrow[r, "\theta^{gp}"] & Q^{gp} \end{tikzcd}$$

   is Cartesian,

3. *locally exact if for every face G of Q, the localized map*

$$P_{\theta^{-1}(G)} \to Q_G$$

   is exact,

4. *very locally exact if for every face G of Q such that $\theta^{-1}(Q) = P^*$, the map*

$$P \to Q_G$$

   is exact.

Remark 2.2 If $P = \mathbf{N}$, then θ is exact if and only if it is local. In general, an exact morphism is easily seen to be local, so a locally exact morphism is exact if and only if it is local. Furthermore, if θ is exact and F is any face of P, then the composite $P_F \to Q_F$ is again exact, hence local, and it follows that there is a face G of Q such that $\theta^{-1}(G) = F$. Thus if θ is exact, $\mathrm{Spec}(\theta)$ is surjective. For fine saturated monoids the converse is also true: if P and Q are fine saturated monoids, then θ is exact if and only if $\mathrm{Spec}\,\theta$ is surjective [6, A.3.2.1]. Moreover, if P and Q are fine sharp monoids and

$\theta\colon C_P \to C_Q$ is injective, then θ is locally exact if and only if the map on monoid algebras: $\mathbf{Z}[C_P] \to \mathbf{Z}[C_Q]$ is flat. One direction follows, for example, from the Theorem 2.3 below. See [10] for the other direction as well as the relation between local exactness to Kato's notion of *integrality* for morphisms of monoids.

Theorem 2.3 *Let P and Q be fine monoids and let $\theta\colon C_P \to C_Q$ be an injective and local homomorphism of their corresponding real cones. Assume P is sharp. Then the following conditions are equivalent.*

1. *θ is locally exact.*
2. *θ is very locally exact.*
3. *For each $q \in C_Q$,*
$$S_q := (q + \theta(C_P^{gp})) \cap C_Q$$
is isomorphic as a C_P-set to C_P, generated by a unique $q_0 \in C_Q$ with the property that $\theta^{-1}\langle q_0\rangle = \{0\}$.

Proof: In what follows we regard C_P as a subcone of C_Q. Observe first that, in any case, $\{S_q : q \in C_Q\}$ forms a partition of C_Q. We claim that each S_q contains some q_0 such that $\langle q_0\rangle \cap C_P = \{0\}$. Choose a local homomorphism $h\colon C_Q \to (\mathbf{R}_\geq, +)$. Then h factors through a local homomorphism $\overline{h}\colon C_{\overline{Q}} \to (\mathbf{R}_\geq, +)$, where $\overline{Q} := Q/Q^*$ is sharp. One verifies immediately that the image of S_q in $C_{\overline{Q}}$ is exactly $S_{\overline{q}}$. Since $\overline{Q}$ is sharp, $\overline{h}^{-1}[0, \overline{h}(\overline{q})]$ is compact, and since $S_{\overline{q}}$ is closed in $C_{\overline{Q}}$, its intersection with $\overline{h}^{-1}[0, \overline{h}(\overline{q})]$ is also compact. It follows that $\overline{h}$ achieves a minimum value on $S_{\overline{q}}$, say at $\overline{q}_0$, where $q_0 \in S_q$. Suppose that $p \in \langle q_0\rangle \cap C_P$, so that there exists some $q' \in C_Q$ such that $q_0 = q' + p$. Then $q' \in S_{q_0} = S_q$, and $h(q_0) = h(q') + h(p) \geq h(q')$. Since $h(q') \geq h(q_0)$, it follows that $h(p) = 0$, and since P is sharp, it follows that $p = 0$. Thus $\langle q_0\rangle \cap C_P = \{0\}$ as desired. Now if (2) holds, the map $C_P \to C_Q$ remains exact after localizing by q_0. Since $q_0 \in S_q$, there exist p and $p' \in C_P$ such that $q + p = q_0 + p'$. Thus $p' - p$ lies in the localization of C_Q by q_0, and hence $p' - p \in C_P$. It follows that $q = p' - p + q_0$ lies in the C_P orbit of q_0. Since $q \in S_q$ was arbitrary, we see that q_0 generates S_q as a C_P-set. Now if q_1 is another element of S_q such that $\langle q_1\rangle \cap C_P = \{0\}$, we have $q_1 = q_0 + p$ and $q_0 = q_1 + p'$ for some $p, p' \in C_P$. Since C_Q is integral, this implies that

$p + p' = 0$, and since P is sharp, it follows that $p = p' = 0$ and that $q_0 = q_1$. This proves that (2) implies (3).

Now suppose that (3) holds. Let G be a face of Q and let F be its intersection with P. We will prove that $(C_P)_F \to (C_Q)_G$ is exact. Suppose that $p_2 - p_1$ lies in the localization of C_Q by G. Then there exist $q \in Q$ and $g \in G$ such that $p_2 - p_1 = q - g$. This implies that $S_g = S_q$. Condition (3) implies that there exists some $q_0 \in C_Q$ which generates $S_g = S_q$ as a C_P-set. Thus there are $p, p' \in C_P$ such that $g = p + q_0$ and $q = p' + q_0$. Then $p_2 - p_1 = p' - p$. On the other hand, $p \in \langle g \rangle \subseteq G$, so $p \in F$, so indeed $p_2 - p_1 \in (C_P)_F$, as required. This proves that (3) implies (1). The remaining implication is trivial. $\square$

The following consequence is a key ingredient in our proof of relative rounding. See [6, A.3.2.2].

Corollary 2.4 *Let P and Q be fine monoids and let $\theta\colon C_P \to C_Q$ be a local and locally exact morphism of the corresponding cones. Assume that P is sharp, and let $C_{Q,P} \subseteq C_Q$ denote the union of the set of faces G of C_Q such that $G \cap C_P = \{0\}$. Then the map*

$$g\colon C_{Q,P} \times C_P \to C_Q : (q, p) \mapsto q + p$$

is a homeomorphism, with inverse

$$(\pi_{Q,P}, \pi_P)\colon C_Q \to C_{Q,P} \times C_P.$$

Proof: Since θ is local and locally exact, it is exact and therefore injective, since P is sharp. It follows immediately from condition (3) of Theorem 2.3 that the map g is bijective. It is clear that g is continuous, and to prove that it is a homeomorphism, one can use Lemma 1.6. It will suffice to prove that if B is a subset of $C_{Q,P} \times C_P$ whose image B' in C_Q is bounded, then B is also bounded. Let G be a face of C_Q which intersects C_P trivially, and let $h\colon C_Q \to (\mathbf{R}_{\geq}, +)$ be a homomorphism such that $h^{-1}(0) = C_G$. Let B_G be the subset of B consisting of those pairs (q, p) such that $q \in G$ and let B'_G be its image. If $(q, p) \in B_G$, $h(p) = h(q + p) \subseteq h(B'_G)$, and since $B'_G \subseteq B'$ is bounded, so is $h(B'_G)$. Since h restricts to a local homomorphism on C_P and P is sharp, it follows that the projection of B_G to C_P is bounded. Repeating this for each face G of Q with $G \cap C_P = 0$, we conclude that the projection of B to C_P is bounded. Since we also know that B' is bounded, it follows that the projection of B to $C_{Q,P}$ is also bounded. Hence B is bounded. $\square$

Our strategy for the proof of Theorem 0.2 will be to use moment maps to compare the spaces X_Q and X_P to the corresponding cones C_Q and C_P and then apply Corollary 2.4. This is not straightforward because the moment map is not functorial. Overcoming this difficulty is the main content of the rest of this section.

If P is any monoid, let $P^+ := P \setminus P^*$ denote its maximal ideal. The *vertex* of X_P is the element of X_P defined by

$$v_P(p) := \begin{cases} 0 & \text{if } p \in P^+ \\ 1 & \text{if } p \in P^* \end{cases} .$$

If $\theta\colon C_P \to C_Q$ is a morphism of cones, let $X_{Q,P} := X_\theta^{-1}(Z(P^+))$, *i.e.*, the Zariski closed subset of X_Q defined by the ideal of C_Q generated by $\theta(P^+)$. Let J be the radical of this ideal. Then $X_{Q,P} = Z(J)$, and J is the intersection of the prime ideals of C_Q containing C_P^+, *i.e.*, the prime ideals corresponding to the faces G of C_Q such that $\theta^{-1}(G) = C_P^*$ (see Remark 1.1). When P is sharp, $X_{Q,P} = X_\theta^{-1}(v_P)$.

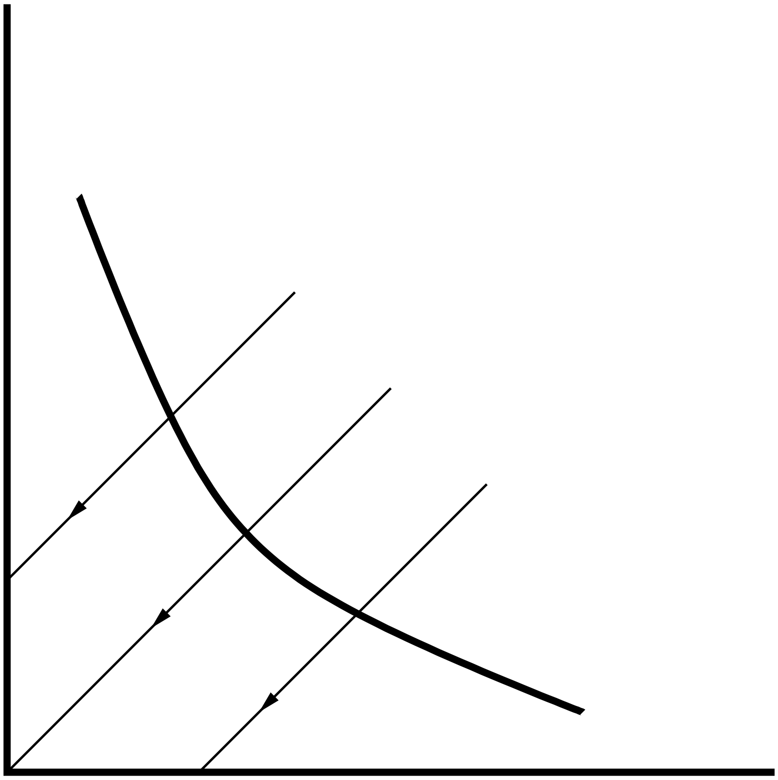
Let A be an effective cycle in C_Q whose support generates C_Q, and let $\mu_A\colon X_Q \to C_Q$ be the associated moment map (1.2). We know from Theorem 1.4 that μ_A is a homeomorphism compatible with the stratifications by faces. Then its inverse ν_A enjoys the same property. Thus $\mu_A(X_{Q,P}) = C_{Q,P} \subseteq C_Q$ and ν_A induces a homeomorphism $\nu_{A,P}\colon C_{Q,P} \to X_{Q,P}$. We obtain a map $X_Q \to X_{Q,P}$ which will allow us to push the general fibers of X_θ to the special fiber. Figure 1 illustrates this map when θ is the diagonal embedding $\mathbf{N} \to \mathbf{N} \oplus \mathbf{N}$ and A is the minimal set of generators of $\mathbf{N} \oplus \mathbf{N}$; in this case μ_A is the identity map of $\mathbf{R}_\geq \times \mathbf{R}_\geq$.

Theorem 2.5 *Let P and Q be fine monoids, with P sharp, and let $\theta\colon C_P \to C_Q$ be a local and locally exact morphism of their corresponding real cones. Let A be an effective generating cycle of C_Q and let $\eta_{A,P}$ be the composite:*

$$\eta_{A,P} : X_Q \xrightarrow{\mu_A} C_Q \xrightarrow{\pi_{Q,P}} C_{Q,P} \xrightarrow{\nu_{A,P}} X_{Q,P},$$

where $\pi_{Q,P}\colon C_Q \to C_{Q,P}$ is the map defined in Corollary 2.4. Then in the

Figure 1:



commutative diagram

$$\begin{tikzcd}
X_Q \arrow[r, "(\eta_{A,P}, X_\theta)"] \arrow[dr, "X_\theta"'] & X_{Q,P} \times X_P \arrow[d, "pr_2"] \\
& X_P,
\end{tikzcd}$$

the horizontal arrow $(\eta_{A,P}, X_\theta)$ *is a homeomorphism.*

Remark 2.6 The map $\eta_{A,P}\colon X_Q \to X_{Q,P}$ has the following property. For each $x \in X_Q$, $F(x)$ is the face of C_Q generated by $F(\eta_{A,P}(x))$ and $F(x) \cap C_P$, and, consequently, the face generated by $F(x)$ and C_P is the same as the face generated by $F(\eta_{A,P}(x))$ and C_P. To check this, let $x' := \eta_{A,P}(x)$, so that $\mu_A(x) = \mu_A(x') + p$ for some $p \in C_P$. This implies that p and $\mu_A(x')$ belong to $F := \langle \mu_A(x) \rangle$, and hence that F is the face of C_Q generated by $\mu_A(x')$ and $F \cap C_P$. Now recall that $F(x)$ is the face of C_Q generated by $\mu_A(x)$ and similarly for x'.

Proof of Theorem 2.5 Fix an element q of Q and let $\rho_{P,q}\colon C_P \to X_P$ be the composite

$$\rho_{P,q}\colon C_P \xrightarrow{p \mapsto q+\theta(p)} C_Q \xrightarrow{\nu_A} X_Q \xrightarrow{X_\theta} X_P. \tag{3}$$

Let us note that if F is a face of P, then θ induces a map $C_F \to C_Q$, and we also find a map $\rho_{F,q}\colon C_F \to X_F$. The following shows that, with suitable hypotheses, the maps $\rho_{P,q}$ and $\rho_{F,q}$ are compatible.

Lemma 2.7 *Let P and Q be fine monoids and let $\theta\colon C_P \to C_Q$ be an injective homomorphism of their corresponding cones. Suppose that $q \in C_Q$ is contained in a face G of C_Q such that $G^{gp} \cap P^{gp} = 0$ and such that the map $C_P \to C_Q/G$ is exact. Then the map (3): $\rho_{P,q}\colon C_P \to X_P$ is compatible with the stratifications by faces: it sends each face C_F of C_P to the corresponding* *subset X_F of X_P, and there is a commutative diagram:*

$$\begin{tikzcd}
 & X_F \arrow[d, "i_F"] \\
C_F \arrow[ur, "\rho_{F,q}"] \arrow[r, "\rho_{P,q_{|C_F}}"] \arrow[dr, "\rho_{F,q}"'] & X_P \arrow[d] \\
 & X_F
\end{tikzcd}$$

Proof: Since $C_P \to C_Q/G$ is exact, each face F of C_P lifts to a face of C_Q/G. Thus there exists a face F' of C_Q containing G such that $F' \cap C_P = F$. Choose a homomorphism $h\colon C_Q \to (\mathbf{R}_{\geq}, +)$ with $h^{-1}(0) = F'$. If $p \in F$ and $x := \nu_A(q+p)$,

$$0 = h(q+p) = \sum_s a_s x(s) h(s).$$

Since $a_s > 0$ for all s in the support of A and $h(s) \neq 0$ if $s \notin F'$, it follows that $x(s) = 0$ if $s \notin F'$. This means that $x \in X_{F'} \subseteq X_Q$, and hence that $X_\theta(x) \in X_F \subseteq X_P$. $\square$

Before proceeding, let us review some elementary compatibilities about bilinear forms in our context. Let V be a finite dimensional real vector space, let $V^\vee$ be its dual, and let $\tau\colon V^\vee \to V$ be a linear map. Recall that the associated bilinear form β on $V^\vee$ is defined by

$$\beta(\phi, \psi) := \psi(\tau(\phi)).$$

Symmetry of β means that the diagram

$$\begin{tikzcd}
V^\vee \arrow[r, "\tau"] \arrow[d, "\mathrm{id}"'] & V \arrow[d, "ev"] \\
V^\vee \arrow[r, "\tau^\vee"] & V^{\vee\vee}
\end{tikzcd}$$

commutes. Assume now that β is symmetric and positive definite. Then τ is an isomorphism and the bilinear form on V corresponding to the map $\tau^{-1}\colon V \to V^\vee$ is just the bilinear form obtained from β by transport of structure using the isomorphism τ. Let us denote this bilinear form (which is also positive definite and symmetric) also by β. Note that if W is a linear subspace of V, β restricts to a positive definite bilinear form on W and hence defines an isomorphism $W \to W^\vee$. In fact the following diagram also commutes:

$$\begin{tikzcd}
V^\vee \arrow[d] & V \arrow[l, "\tau^{-1}"'] \\
W^\vee & W. \arrow[l, "\tau^{-1}|_W"'] \arrow[u]
\end{tikzcd} \tag{4}$$

Let F be a face of P. By Lemma 2.7, $\rho_{P,q}$ induces a map $C_F \to X_F \subseteq X_P$, which we can identify with $\rho_{F,q}$, and we just write ρ_q in either case. Note that the map $+_q : p \mapsto q + \theta(p)$ maps C_F into the face C_G of C_Q generated by $\theta(F)$ and q and induces a map $C_F^o \to C_G^o$. To simplify the notation, we replace Q by G. If $p \in F$, let $x := \rho_q(p)$. Then we have commutative diagrams:

$$\begin{tikzcd}
X_Q^* \arrow[d, "X_\theta"'] & C_Q^o \arrow[l, "\mu_A^{-1}"'] & & Lie(X_Q^*) \arrow[d] & Lie(C_Q^o) \arrow[l, "\tau_x^{-1}"'] \\
X_F^* & C_F^o \arrow[l, "\rho_q"'] \arrow[u, "+_q"'] & & Lie(X_F^*) & Lie(C_F^o). \arrow[l, "\tau_x^{-1}"'] \arrow[u]
\end{tikzcd} \tag{5}$$

Here the diagram on the right is the derivative of the diagram on the left, and identifies with diagram (4). Thus $d\rho_q$ at p can be identified with τ_x^{-1}.

Lemma 2.8 *Let $q \in C_Q$ be as in Lemma 2.7 and let F be a face of P. The restriction of ρ_q to the interior of C_F^o is a homeomorphism onto its image, which is an open subset of $X_F^* \subseteq X_P$.*

Proof: It follows from diagrams (5) and (4) that if $p \in C_F^o$, the derivative of the map $\rho_q\colon C_F \to X_F$ at p is the isomorphism $V_F \to V_F^\vee$ corresponding to the restriction of the bilinear form $\beta_{\rho_q(p)}$ on V_Q to V_F. By the implicit function theorem, the restriction of ρ_q to C_F^o is an isomorphism locally on C_F^o, and in particular its image is an open subset of X_F^*.

Let p and p' be two distinct elements of C_F^o, let $v := p' - p \in V_F$, and for $t \in [0,1]$ let $f(t) := p + tv$. Then f is a continuous map from $[0,1]$ to C_F^o, and its derivative at any $t \in (0,1)$ is v. The logarithm map induces an isomorphism $X_F^* \cong \operatorname{Hom}(F^{gp}, (\mathbf{R}, +))$, and evaluation at v defines a map $\log(e_v) \colon X_F^* \to (\mathbf{R}, +)$. Recall that $d\log(e_v)$ is the invariant differential form corresponding to the element $v \in V \cong Lie(X_F^*)^\vee$, and that the derivative of the map $C_F^o \to X_F^*$ is the isomorphism corresponding to β_{ρ_q}. It follows that the derivative of the composite $\log(e_v) \circ \rho_q \circ f$ is $\beta_{\rho_q \circ f}(v, v)$. Since β is everywhere positive definite, this function is increasing. Hence $\log(e_v)(\rho_q(p)) \neq \log(e_v)(\rho_q(p'))$, so $\rho_q(p) \neq \rho_q(p')$, as required. □

Lemma 2.9 *Let (q_n, p_n) be a sequence in $C_Q \times C_P$. If (q_n) is bounded and (p_n) is unbounded, then $\rho_{q_n}(p_n)$ is unbounded.*

Proof: Choose any norm on V_P, and let $\lambda_n := ||p_n||$. Replacing (q_n, p_n) by a subsequence, we may assume that $\lambda_n \neq 0$ for all n and that the sequence (λ_n) tends to infinity. Let $p'_n := p_n/\lambda_n$. Then p'_n is a sequence in the unit ball of V_P, and hence contains a convergent subsequence. Passing to this subsequence, we may assume that (p'_n) converges to some p' in the unit ball. Let $x_n := \nu_A(q_n + p_n)$. Let S be the support of A, let S_u be the set of all $s \in S$ such that $x_n(s)$ is unbounded and let S_b be its complement. Then

$$\lambda_n^{-1} q_n + p'_n = \lambda_n^{-1} \mu_A(x_n) = \sum_{s \in S_u} \lambda_n^{-1} a_s x_n(s) s + \sum_{s \in S_b} \lambda_n^{-1} a_s x_n(s) s. \tag{6}$$

The second sum on the right side converges to zero, and it follows that the first sum converges to p'. Since $p' \neq 0$, S_u is not empty. Choose some $s_1 \in S_u$, and let (n_k) be an increasing sequence such that the subsequence $(x_{n_k}(s_1))$ tends to infinity. Let S'_u be the set of all s such that $x_{n_k}(s)$ is unbounded and let S'_b be its complement. Then equation (6) holds with the subsequence (q_{n_k}, p_{n_k}) in place of (q_n, p_n) and with S_u and S_b replaced by S'_u and S'_b. Furthermore $s_1 \in S'_b \subseteq S_b$. After repeating this process a finite number of times, we find a subsequence of the original sequence with the property that $S = S_u \cup S_b$, where $x_n(s)$ approaches infinity if $s \in S_u$ and is bounded if $s \in S_b$. Equation (6) still holds, and hence p' is contained in the closure of the subcone C_{S_u} of C_P spanned by S_u. Since S_u is finite, this cone is already closed, and so in fact $p' \in C_{S_u}$. Write $p' = \sum\{c_{s'} s' : s' \in S_u\}$, where $c_{s'} \geq 0$. Then $x_n(p') = \prod x_n(s')^{c_{s'}}$. Since each sequence $x_n(s')$ tends to infinity and at least one $c_{s'}$ is positive, $x_n(p')$ is unbounded. Since $p' \in C_P$, it follows that $X_\theta(x_n)$ is unbounded in X_P. □

Lemma 2.10 *For each $q \in C_Q$ as in Lemma 2.7, the map $\rho_q\colon C_P \to X_P$ is surjective.*

Proof: Let F be a face of P. We shall prove that the map $\rho_q\colon C^o_F \to X^*_F$ is surjective. We may as well take $F = P$. Choose a generating effective cycle B of C_P and recall that the moment map $\mu_B\colon X^*_P \to C^o_P$ is injective. Thus it suffices to prove that for each $c \in C^o_P$, there is a point $p \in C_P$ such that $\mu_B \circ \rho_q(p) = c$. We follow the method of the proof of the surjectivity of moment mapping in Theorem 1.4. Consider the function

$$\Psi_c = e_c \exp(-e_B)\colon X_P \to C_P,$$

where $e_B := \sum b_p e_p$. This function is nonzero on X^*_P and $d \log \Psi_c = c - \mu_B$ on X^*_P. Thus it suffices to show that there is some point x in the image U^*_q of the restriction of ρ_q to C^o_P at which $d \log \Psi_c = 0$. We know already that U^*_q is open in X^*_P, so it suffices to show that Ψ_c has a critical point somewhere on U^*_q. Thus it will suffice to show that Ψ_c has a maximum somewhere on U^*_q, or equivalently, that $\Psi_c \circ \rho_q$ has a maximum somewhere on C^o_P. For each $r > 0$, let $C_{B,c}(r)$ denote the set of points of C_P where $\Psi_c \circ \rho_q$ is at least r, and let $X_{B,c}(r)$ denote the set of points of X_P where Ψ_c is at least r. Note that since Ψ_c vanishes on $X_P \setminus X^*_P$, $C_{B,c}(r)$ is contained in C^o_P. Lemma 1.5, applied to P, B, and c, implies that $X_{B,c}(r)$ is bounded. Since $\rho_q(C_{B,c}(r)) \subseteq X_{B,c}(r)$, Lemma 2.9 implies that $C_{B,c}(r)$ is also bounded, hence compact. Choose some $p_0 \in C^o_P$, and let $r := 1/2\Psi_c(\rho_q(p_0))$. Then $\Psi_c \circ \rho_q$ has a maximum on $C_{B,c}(r)$, which will be a maximum of $\Psi_c \circ \rho_q$ on all of C^o_P, as required. $\square$

We can now finish the proof of Theorem 2.5. From Theorem 1.4 and Corollary 2.4 we have homeomorphisms

$$\mu_A\colon X_Q \to C_Q \quad \text{and} \quad g\colon C_{Q,P} \times C_P \to C_Q.$$

Thus it will suffice to show that $h := (\eta_{A,P}, X_\theta) \circ \mu_A^{-1} \circ g$ is a homeomorphism. This is the map

$$h\colon C_{Q,P} \times C_P \to X_{Q,P} \times X_P : (q,p) \mapsto (\mu_A^{-1}(q), \rho_q(p)),$$

It is clear that h is continuous. Since μ_A is a bijection $X_{Q,P} \to C_{Q,P}$ and since each ρ_q is bijective, it follows that h is also bijective. To prove that it is a homeomorphism it will suffice to show that h^{-1} takes bounded sets to bounded sets, by Lemma 1.6. Suppose that (q_n, p_n) is a sequence in $C_{Q,P} \times C_P$ such that $(h(q_n, p_n))$ is bounded. Then $(\mu_A^{-1}(q_n))$ is bounded, hence (q_n) is bounded. Since $(\rho_{q_n}(p_n))$ is bounded, it follows from Lemma 2.9 that (p_n) is also bounded. $\square$

As an offshoot of our technique, we can force the following functoriality for moment maps.

Proposition 2.11 *Let Q be a fine monoid and let $\theta\colon P \to Q$ be the inclusion of an exact submonoid. Let S (resp. T) be a finite set of generators for Q (resp. P). Then*

$$X_\theta \circ \mu_S^{-1} \circ C_\theta \colon C_P \to X_P$$

is a homeomorphism which sends each face of C_P to the corresponding face of X_P. Hence there is a face-preserving homeomorphism $h_{S,T}\colon X_P \to X_P$ which makes the following diagram commute:

$$\begin{tikzcd}
C_Q \arrow[r, "\mu_S^{-1}"] & X_Q \arrow[dr, "X_\theta"] & \\
C_P \arrow[u, "C_\theta"] \arrow[r, "\mu_T^{-1}"] & X_P \arrow[r, "h_{S,T}"] & X_P
\end{tikzcd}$$

Proof: The proof of Theorem 2.5 shows that ρ_0 is a homeomorphism when $\theta\colon P \to Q$ is the inclusion of an exact submonoid. Then $h_{S,T} := \rho_0 \circ \mu_T$ fits into the diagram shown. $\square$

Proof of Theorem 0.2 The only difficulty is that in Theorem 0.2, the monoid P need not be sharp. To reduce to Theorem 2.5, let $\overline{Q} := Q/Q^*$ and $\overline{P} := P/P^*$ and consider the diagram of finite dimensional vector spaces:

$$\begin{tikzcd}
C_{Q^*} \arrow[r] & C_Q^{gp} \arrow[r] & C_{\overline{Q}}^{gp} \\
C_{P^*} \arrow[u, "\theta^*"] \arrow[r] & C_P^{gp} \arrow[u, "\theta^{gp}"] \arrow[r] & C_{\overline{P}}^{gp}. \arrow[u, "\overline{\theta}^{gp}"]
\end{tikzcd}$$

Note that since θ is local, the map $\overline{\theta}^{gp}$ is injective. Choose a splitting $C^{gp}_Q \to C^*_Q$ which maps C^{gp}_P to C^*_P. Note that the corresponding splittings $C^{gp}_{\overline{Q}} \to C^{gp}_Q$ and $C^{gp}_{\overline{P}} \to C^{gp}_P$ necessarily map $C_{\overline{Q}}$ to C_Q and $C_{\overline{P}}$ to C_P, respectively. It follows that there is a commutative diagram:

$$\begin{tikzcd}
X_Q \arrow[r, "\cong"] \arrow[d, "X_\theta"'] & X_{Q^*} \times X_{\overline{Q}} \arrow[d, "X_{\theta^*} \times X_{\overline{\theta}}"] \\
X_P \arrow[r, "\cong"] & X_{P^*} \times X_{\overline{P}}.
\end{tikzcd} \tag{7}$$

Since $C_{P^*} \to C_{Q^*}$ is an injective map of vector spaces, it admits a splitting, and hence the map X_{θ^*} is isomorphic to a projection map. Thus X_θ will be isomorphic to a projection if $X_{\overline{\theta}}$ is. Since θ is local, the map $\overline{\theta}^{gp}$ is injective. Then $\overline{\theta}$ is a locally exact and injective map of sharp cones, and it suffices to apply Theorem 2.5 to $\overline{\theta}$. $\square$

Theorem 2.5 implies that (with the hypothesis there) all the fibers of the map $X_Q \to X_P$ are homeomorphic. For each point y of X_P, let $X_Q(y)$ denote the fiber of X_θ over y. In particular, the "broken" fiber $X_Q(v) = X_{Q,P}$ over the vertex v of X_P is homeomorphic to the fiber $X_Q(1)$ over 1, where 1 is the identity element of the monoid X_P. Explicitly, if $y \in X_Q(v)$, there is a unique $s(y) \in X_Q(1)$ such that $\eta_{A,P}(s(y)) = y$, and $s\colon X_Q(v) = X_{Q,P} \to X_Q(1)$ is a homeomorphism. Let

$$\eta'_{A,P} := s \circ \eta_{A,P}\colon X_Q \to X_Q(1).$$

We obtain the following reformulation of Theorem 2.5, which is sometimes more convenient.

Corollary 2.12 *With the hypotheses of Theorem 2.5, there is a continuous map $\eta'_{A,P}\colon X_Q \to X_Q(1)$ with the following properties:*

1. *The map $(\eta'_{A,P}, X_\theta)\colon X_Q \to X_Q(1) \times X_P$ is a homeomorphism, and the*

diagram

$$\begin{tikzcd}
X_Q \arrow[r, "(\eta'_{A,P}, X_\theta)"] \arrow[dr, "X_\theta"'] & X_Q(1) \times X_P \arrow[d, "pr_2"] \\
& X_P
\end{tikzcd}$$

is commutative.

2. *For each $x \in X_Q$, $F(\eta'_{A,P}(x))$ is the face of Q generated by $F(x)$ and P.*

Proof: Let x be a point in X_Q, and let $y := \eta_{A,P}(x)$ and $y' := \eta'_{A,P}(x) = s(y)$. Then $\eta_{A,P}(y') = y$, and it follows from Remark 2.6 that

$$\langle F(x) + P \rangle = \langle F(y) + P \rangle = \langle F(y') + P \rangle.$$

Since $y' \in X_Q(1)$, $F(y')$ already contains P, so $F(\eta'_{A,P}(x))$ is the face generated by $F(x)$ and P. $\square$

Remark 2.13 Let us attempt to describe the fibers of the map X_θ in Corollary 2.12. The fiber $X_Q(1)$ consists of the set of homomorphisms $x\colon C_Q \to (\mathbf{R}_\geq, \cdot)$ sending C_P to 1, or, equivalently, the set of homomorphisms $C_Q/C_P \to (\mathbf{R}_\geq, \cdot)$, where C_Q/C_P is the image of C_Q in the quotient C_Q^{gp}/C_P^{gp}. Note that this identification $X_Q(1) \cong X_{C_Q/C_P}$ is compatible with the stratification by faces: if $x \in X_Q(1)$ corresponds to $x' \in X_{C_Q/C_P}$, then $F(x)$ is the face of C_Q containing C_P corresponding naturally to the face $F(x')$ of C_Q/C_P. We will allow ourselves to identify the faces of C_Q/C_P with those faces of C_Q containing C_P when convenient.

A morphism of monoids or cones $\theta\colon P \to Q$ is said to be *dominating* or *vertical* if the image Q/P of Q in $\mathrm{Cok}(\theta^{gp})$ is a group, or equivalently if the image of P is not contained in any proper face of Q. If $\theta\colon C_P \to C_Q$ is vertical, then $X_Q(1) \subseteq X_Q^*$ and in fact $X_Q(1)$ is a smooth submanifold of X_Q^*. Indeed, $X_Q(1) \cong X_{C_Q/C_P}$ and C_Q/C_P is a group, so that in fact X_{C_Q/C_P} is homeomorphic to a Euclidean space. More specifically, if $x \in X_Q$ we say that X_θ is *vertical at* x if the localization of Q by $F(x) + P$ is a group, or equivalently, if $\langle F(x) + P \rangle = Q$. It is clear that the set X_Q^v of all x at which X_θ is vertical is an open subset of X_Q.

The following result describes the fibers of the submersions arising in Theorem 2.5.

Proposition 2.14 *Let Q and P be fine monoids.*

1. *There exist a manifold with boundary (M, B) and homeomorphisms:*

   $$(M, M \setminus B) \cong (X_Q, X_Q^*) \cong (C_Q, C_Q^o).$$

   In particular, the boundary B is empty if and only if Q is a group.

2. *Let $\theta\colon C_P \to C_Q$ be an exact, injective, and locally exact morphism of fine monoids, and let $X_Q^v \subseteq X_Q$ denote the open subset of points at which X_θ is vertical. Then there is a commutative diagram*

   $$\begin{tikzcd}
   (X_Q, X_Q^v) \arrow[r, "\cong"] \arrow[dr, swap, "X_\theta"] & (X_{Q/P}, X^*_{Q/P}) \times X_P \arrow[d, "pr_2"] \\
   & X_P
   \end{tikzcd}$$

   in which the top horizontal arrow is a homeomorphism. In particular, the fibers of $X_Q \to X_P$ are all topological manifolds with boundary, and the boundary is empty if and only if θ is vertical.

Proof: Although statement (1) is well-known, we include a proof for the sake of completeness. The moment map associated with any effective generating cycle induces a homeomorphism

$$(X_Q, X_Q^*) \cong (C_Q, C_Q^o)$$

so it suffices to prove that (C_Q, C_Q^o) is homeomorphic to some $(M, M \setminus B)$. Choose a splitting $C_Q \cong C_{Q^*} \times C_{\overline{Q}}$. Then

$$(C_Q, C_Q^o) \cong (C_{Q^*} \times C_{\overline{Q}}, C_{Q^*} \times C^o_{\overline{Q}}),$$

and C_{Q^*} is a Euclidean space. Hence it suffices to prove the result when Q is sharp. Choose an element p of the interior of Q and let P be the submonoid of Q generated by p. Then $P \to Q$ is exact and locally exact, and the quotient Q/P is a group. It follows from Corollary 2.4 that the projection mapping $C_Q \to C_{Q/P}$ induces a homeomorphism $C_{Q,P} \to C_{Q/P}$, and then a homeomorphism $C_{Q/P} \times C_P \to C_Q$, sending $C_{Q/P} \times C_P^o$ to C_Q^o. Alternatively one can argue from Corollary 2.12. As we saw in Remark 2.13, $X_Q(1) \cong X_{Q/P}$ is homeomorphic to a Euclidean space $\mathbf{R}^{n-1}$, where n is the dimension of C_Q. Thus $X_Q(1) \times \mathbf{R}_\geq$ is a manifold with boundary $X_Q(1) \times \{0\}$. Since $X_\theta^{-1}(\mathbf{R}^+) = X_Q^*$, the homeomorphism of Corollary 2.12 produces the desired homeomorphism

$$(X_Q, X_Q^*) \cong (X_Q(1) \times \mathbf{R}_\geq, X_Q(1) \times \mathbf{R}^+).$$

The horizontal arrow in the diagram of statement (2) is the same as that of the diagram in statement (1) of Corollary 2.12. Statement (2) of this corollary implies that this arrow maps X_Q^v to $X_{Q/P}^*$. Statement (1) of the proposition, applied to the fine monoid Q/P, tells us that $(X_{Q/P}, X_{Q/P}^*)$ is a topological manifold with boundary. $\square$

3 Submersivity of log smooth maps

The main result of this section is Theorem 3.5 below. Before stating it, we shall review the notion of a log analytic space X and its associated topological space X_{log}. Let us begin with a version for toric varieties, which serve as local models for smooth log analytic spaces.

Let Q be a fine monoid and let A_Q denote the set of homomorphisms of $Q \to (\mathbf{C}, \cdot)$, *i.e.*, the set of complex points of Spec $\mathbf{C}[Q]$. We endow A_Q with its natural structure of a complex analytic variety, which is the analytic space associated to an affine toric variety.[1] As alluded to in the introduction, there is a natural way to introduce polar coordinates for such varieties. Recall that $X_Q := \operatorname{Hom}(Q, (\mathbf{R}_\geq, \cdot))$, which is naturally a subset of A_Q. Similarly we define $T_Q := \operatorname{Hom}(Q, \mathbf{S}^1) \subseteq A_Q$, where $\mathbf{S}^1 := \{z \in \mathbf{C} : |z| = 1\}$. Then T_Q acts naturally on A_Q by multiplication. If $x \in X_Q$, the isotropy subgroup of T_Q at x is $T_{Q/F(x)} \subseteq T_Q$. Furthermore, there is a natural retraction $A_Q \to X_Q$, sending a point z to the point $|z|$. The following proposition is then immediate.

Proposition 3.1 *Let*

$$\tau_Q \colon A_Q^{log} := T_Q \times X_Q \to A_Q$$

---

[1] Strictly speaking A_Q is traditionally called a toric variety only when Q is fine and saturated and Q^{gp} is torsion free.

be the map induced by the action of T_Q *on* A_Q *and the inclusion* $X_Q \to A_Q$*. Then* τ_Q *is surjective and proper, and for* $z \in A_Q$*,* $\tau_Q^{-1}(z) \cong T_{Q/F(|z|)}$*.*

□

We can now state the following local version of Theorem 3.5.

Proposition 3.2 *Let* $\theta\colon P \to Q$ *be a locally exact and injective homomorphism of fine monoids. Then the map* $A_\theta^{log}\colon A_Q^{log} \to A_P^{log}$ *is submersive. Its fibers are topological manifolds with boundary, and the boundary consists of those points of* A_Q^{log} *lying over points of* X_Q *at which* X_θ *is not vertical (see Remark 2.13).*

Proof: Let $F := \theta^{-1}(Q^*)$. Then F is a face of P and θ factors: $\theta = \theta' \circ \theta''$, where $\theta''\colon P \to P_F$ and $\theta'\colon P_F \to Q$. Since the composition of two submersive maps is submersive, it suffices to prove the result for θ' and θ''. The map $A_{\theta''}\colon A_{P_F} \to A_P$ is an open immersion, and the same is true for $A_{\theta''}^{log}$, and hence it is certainly submersive. The map $\theta'\colon A_{P_F} \to A_Q$ is local and locally exact, hence exact. Thus we are reduced to proving the proposition when θ is exact and locally exact. Since $A_\theta^{log} \cong T_\theta \times X_\theta$ and the product of two submersions is a submersion, it will suffice to prove that T_θ and X_θ are submersive. Theorem 0.2 says that X_θ is a projection mapping, hence submersive.

The proposition will now follow if we prove that T_θ is submersive. This is almost immediate. Consider the exact sequence of finitely generated abelian groups:

$$0 \to P^{gp} \to Q^{gp} \to G \to 0,$$

where G is by definition the quotient of Q^{gp} by the image of P^{gp}. The obstruction ξ to splitting this sequence lies in $Ext^1(G, P^{gp})$. which is a finite group. It follows that there is natural number n such that $n\xi = 0$. This means that the sequence obtained by pushout:

$$\begin{tikzcd}
P^{gp} \arrow[r] \arrow[d, "\cdot n"'] & Q^{gp} \arrow[r] \arrow[d] & G \arrow[d, "\mathrm{id}"] \\
P^{gp} \arrow[r] & \tilde{Q}^{gp} \arrow[r] & G
\end{tikzcd}$$

splits, and hence that the corresponding map $T_{\tilde{\theta}}\colon T_{\tilde{Q}^{gp}} \to T_{P^{gp}}$ is a product map. On the other hand, the square in the diagram

$$\begin{tikzcd}
T_G \times T_{P^{gp}} \arrow[r, "\cong"] \arrow[dr, swap, "pr_2"] & T_{\tilde{Q}^{gp}} \arrow[r] \arrow[d, swap, "T_{\tilde{\theta}}"] & T_{Q^{gp}} \arrow[d, "T_\theta"] \\
& T_{P^{gp}} \arrow[r, "\cdot n"] & T_{P^{gp}}
\end{tikzcd} \tag{8}$$

is Cartesian, and the bottom arrow is a covering space. It follows that T_θ is a fiber bundle, and in particular is submersive.

Finally, let us remark that the fiber of A_θ^{log} over a point (ζ, x) of A_P^{log} is just the product of the fiber of T_θ over ζ and the fiber of X_θ over x. The former is the manifold T_G and the latter is a manifold with boundary as described in Proposition 2.14. This proves the last statement. $\square$

Remark 3.3 The proof shows that if $P \to Q$ is local as well as locally exact then A_θ^{log} is a fiber bundle, and if in addition the cokernel of $P^{gp} \to Q^{gp}$ is torsion free, then in fact A_θ is a trivial fiber bundle, *i.e.*, is isomorphic to a projection mapping.

Let us briefly recall the basic definitions in log geometry. For details we refer to [10] and [6].

Definition 3.4 *A log structure on an analytic space X is a morphism of sheaves of monoids*

$$\alpha \colon M_X \to (\mathcal{O}_X, \cdot)$$

such that the induced map

$$\alpha^{-1}(\mathcal{O}_X^*) \to \mathcal{O}_X$$

is an isomorphism.

A *log analytic space* is an analytic space endowed with a log structure α, and a *morphism of log analytic spaces* is a morphism which is compatible with the log structures in the evident sense. A basic example is the following. Let Q be a monoid and let $\beta \colon Q \to (\mathcal{O}_X, \cdot)$ be a morphism of sheaves of monoids.

Then one can form the pushout in the category of sheaves of monoids:

$$\begin{tikzcd}
\beta^{-1}(\mathcal{O}_X^*) \arrow[r] \arrow[d] & Q \arrow[d, "\tilde{\beta}"] \arrow[dr, "\beta"] & \\
\mathcal{O}_X^* \arrow[r] & \mathcal{M} \arrow[r, "\alpha"] & \mathcal{O}_X.
\end{tikzcd}$$

Then α is a log structure on X, and $\tilde{\beta}$ is a *chart* for α. A log structure or space is said to be *coherent* (resp. *fine*) if locally on X it admits charts given by finitely generated (resp. fine) monoids. In particular, on the analytic space A_Q associated to the toric variety $\operatorname{Spec}\mathbf{C}[Q]$ as above, there is a natural map $Q \to \mathcal{O}_{A_Q}$, which defines a fine log structure $\mathcal{M}_Q \to \mathcal{O}_{A_Q}$ on A_Q. When Q is saturated, $\mathcal{M}_Q$ can be identified with the sheaf of holomorphic functions on A_Q which become invertible when restricted to $A_Q^* := A_{Q^{gp}}$.

A morphism $f\colon X \to Y$ of log analytic spaces is said to be *exact* (resp. *vertical*) *at* $x \in X$, if the map of monoids

$$f^\flat\colon M_{Y,f(x)} \to M_{X,x}$$

is exact (resp. vertical) (see Definition 2.1 and Remark 2.13). We say simply that f is exact (resp. vertical) if it is so at every x. For the definition of a smooth morphism of log spaces, we must refer to [6, 2.2].

Associated to a log analytic space (X, α) is continuous map of topological spaces

$$\tau_X\colon X_{log} \to X$$

which globalizes the polar coordinate construction for toric varieties described above. We present here a slight generalization of the original definition given in [11]; note that our definition does not require the log structure to be coherent. By definition X_{log} is the set of pairs (x, σ), where x is a point of X and σ is a homomorphism of monoids fitting into the following commutative diagram:

$$\begin{tikzcd}
M_{X,x} \arrow[r, "\sigma"] & \mathbf{S}^1 \\
\mathcal{O}_{X,x}^*, \arrow[u, "\alpha_x"] \arrow[ur, "\arg_x"'] &
\end{tikzcd}$$

where $\arg_x(u) := u(x)/|u(x)|$. The map $\tau\colon X_{log} \to X$ sends (x, σ) to x. Corresponding to each section m of M_X on an open subset U of X is a function

$$\arg(m)\colon \tau_X^{-1}(U) \to \mathbf{S}^1\colon (x, \sigma) \mapsto \sigma(m_x).$$

We endow X_{log} with the weak topology defined by the functions τ_X and $\arg(m)$ as m ranges over the local sections of M_X. It follows from the fact that $\mathbf{S}^1$ is a compact and divisible group that the map τ_X is surjective and proper. The fiber of a point x is a torsor under the group $\operatorname{Hom}(\overline{M}_{X,x}, \mathbf{S}^1)$.

Theorem 3.5 *Let $f\colon X \to Y$ be an exact smooth morphism of fine log analytic spaces. Then $f_{log}\colon X_{log} \to Y_{log}$ is a topological submersion. The fibers are topological manifolds with boundary, and the boundary consists of the set of points of X_{log} lying over points of X at which f is not vertical.*

Proof: We make use of the following argument from [6, A.3.3] to reduce to Proposition 3.2. The statement is local on X. Choose a point x of X and let $y := f(x)$. It follows from the smoothness of f that, after replacing X by a neighborhood of x and Y by a neighborhood of y, there exist charts $P \to M_Y$ and $Q \to M_X$ and a morphism $\theta\colon P \to Q$ with the following properties.

1. The monoids P and Q are fine, and $\theta\colon P \to Q$ is injective.
2. The maps $P \to \overline{M}_{Y,y}$ and $\overline{Q} \to \overline{M}_{X,x}$ are isomorphisms.
3. The diagram

$$\begin{tikzcd}
X \arrow[r, "i"] \arrow[dr, "f"'] & Y \times_{A_P} A_Q \arrow[r] \arrow[d, "f'"] & A_Q \arrow[d, "A_\theta"] \\
& Y \arrow[r, "\beta"] & A_P
\end{tikzcd}$$

is commutative, its square is Cartesian, and i is a strict open immersion.

Since P is sharp, the vertex v_P of A_P is the point defined by the maximal ideal P^+ of P, and $\beta(y) = v_P$. Then $X_y := f^{-1}(y)$ is open in $A_\theta^{-1}(v_P)$. Choose any prime ideal $\mathfrak{q}$ of Q such that $\theta^{-1}(\mathfrak{q}) = P^+$ and let $G := Q \setminus \mathfrak{q}$ be the corresponding face of Q. Then $A_G \cong Z(\mathfrak{q}) \subseteq A_\theta^{-1}(v_P)$, and A_G^* is dense in A_G. It follows that there is a point x' in X_y which maps to some point of A^*_G. Then the natural map $Q \to M_{X,x'}$ induces an isomorphism $Q/G \to \overline{M}_{X,x'}$. Since f is exact, the map $M_{Y,y} \to M_{X,x'}$ is exact, and it follows that $P \to Q/G$ is exact. Since this was true for an arbitrary face G of Q lying over the trivial face of P, it follows that θ is very locally exact (see Definition 2.1). By Theorem 2.3, θ is in fact exact and locally exact. By Proposition 3.2, A^{log}_θ is submersive, and hence so is the base changed map f' in the diagram above. Since i is an open immersion, f is also submersive. The statement about the fibers also follows from Proposition 3.2, and this completes the proof. $\square$

Sometimes it is convenient to consider log structures which are not coherent (see for example [13], [3], and [12]). For example, if $M_X \to \mathcal{O}_X$ is a coherent log structure on X, any sheaf of faces $\mathcal{F}$ of M_X is again a log structure, and it is often productive to work with $\mathcal{F} \to \mathcal{O}_X$ even if $\mathcal{F}$ is not coherent. When necessary to clarify with which log structure we are working, we will write $(X, \mathcal{F})$ or $X(\mathcal{F})$ to denote the log space $(X, \mathcal{F} \to \mathcal{O}_X)$, and similarly for $\mathcal{M}$.

The following definitions should be regarded as provisional,

Definition 3.6 *Let $\mathcal{F} \to \mathcal{O}_X$ be a log structure on a complex analytic space X.*

1. *$\mathcal{F}$ is relatively coherent if locally on X there exists a coherent log structure $\mathcal{M}$ containing $\mathcal{F}$ such that $\mathcal{F}$ is locally generated as a sheaf of faces in $\mathcal{M}$ by a finite number of sections of $\mathcal{M}$. (In this case one says the $\mathcal{F}$ is relatively coherent in $\mathcal{M}$.)*

2. *Let Y be a fine log analytic space and let $f\colon X(\mathcal{F}) \to Y$ be a morphism of log analytic spaces. One says that f is relatively smooth if locally on X there exists a fine log structure $\mathcal{M}$ on X in which $\mathcal{F}$ is relatively coherent satisfying the following conditions.*
   - *(a) The composed map $X(\mathcal{M}) \to X(\mathcal{F}) \to Y$ is smooth.*
   - *(b) The stalks of the quotient monoid $\mathcal{M}/\mathcal{F}$ are free monoids.*

Let us remark that in the next theorem, which is so far our main justification for the above definition, the proof of submersivity will not use the condition (b). Our proof will show that, assuming only that (a) holds, then

$f_{log}(\mathcal{F})$ is a submersion whose fibers are locally the product of a manifold with boundary and a space with "toric singularities"; *i.e.*, a space homeomorphic to A_M for some fine monoid M.

Theorem 3.7 *Let Y be a fine log analytic space, let $\mathcal{F} \to \mathcal{O}_X$ be a relatively coherent log structure on an analytic space X, and let*

$$f(\mathcal{F})\colon X(\mathcal{F}) \to Y$$

be an exact and relatively smooth morphism of log analytic spaces. Then the map $f_{log}(\mathcal{F})\colon X_{log}(\mathcal{F}) \to Y_{log}$ is submersive. Furthermore, the fibers of $f_{log}(\mathcal{F})$ are manifolds with boundary, and the boundary consists of those points lying over points of X at which f is not vertical.

Proof: Since the statement is local on X and Y, we may assume that there exists a fine log structure $\mathcal{M}$ on X in which $\mathcal{F}$ is relatively coherent and such that $X(\mathcal{M}) \to Y$ is smooth. For each $x \in X$, $\mathcal{F}_x$ is a face of $\mathcal{M}_x$ and hence is an exact submonoid. By hypothesis, $\mathcal{M}_{Y,f(x)} \to \mathcal{F}_x$ is exact, and since the composite of two exact maps is exact, it follows that $\mathcal{M}_{Y,f(x)} \to \mathcal{M}_x$ is exact. Thus the map of log analytic spaces $X(\mathcal{M}) \to Y$ is smooth and exact.

Since $\mathcal{F}$ is relatively coherent in $\mathcal{M}$, we may assume that $\mathcal{F}$ is generated by a finite number of global sections f_i of $\mathcal{M}$. Furthermore we may assume that there exist charts $P \to \mathcal{M}_Y$ and $Q \to \mathcal{M}$ and a morphism θ as in the proof of Theorem 3.5. We may also assume that each f_i lifts to some $q_i \in Q$. Letting G be the face of Q generated by these q_i, we see that $\mathcal{F}$ is the sheaf of faces of $\mathcal{M}$ generated by the image of $G \to \mathcal{M}$. Let $A_Q(\mathcal{M}_Q)$ (resp. $A_Q(\mathcal{F})$) denote A_Q with the log structure coming from $Q \to \mathcal{O}_{A_Q}$ (resp. the sheaf of faces of $\mathcal{M}_Q$ generated by G). Then we have the following commutative diagram:

$$\begin{tikzcd}
X_{log}(\mathcal{M}) \arrow[r] \arrow[d] & X(\mathcal{M}) \arrow[r, "a"] \arrow[d, "h"] & A_Q(\mathcal{M}_Q) \arrow[d] \\
X_{log}(\mathcal{F}) \arrow[r] \arrow[d] & X(\mathcal{F}) \arrow[r, "b"] \arrow[d, "f(\mathcal{F})"] & A_Q(\mathcal{F}) \arrow[d] \\
Y_{log} \arrow[r] & Y \arrow[r, "c"] & A_P.
\end{tikzcd}$$

Here the map a factors through a strict open immersion into the fiber product $Y \times_{A_P} A_Q(\mathcal{M}_Q)$, and the map h is an isomorphism on underlying analytic spaces. It follows that b also factors through a strict open immersion into $Y \times_{A_P} A_Q(\mathcal{F})$. Then the top arrow in the diagram

$$\begin{tikzcd}
X_{log}(\mathcal{F}) \arrow[r] \arrow[d] & A_Q^{log}(\mathcal{F}) \arrow[d] \\
Y_{log} \arrow[r] & A_P^{log}
\end{tikzcd}$$

factors through an open immersion $X_{log}(\mathcal{F}) \to Y_{log} \times_{A_P^{log}} A_Q^{log}(\mathcal{F})$. (Here we use a superscript *log* instead of a subscript for typographical reasons.) Thus we are reduced to the case when $X = A_Q$, $Y = A_P$, and $f \circ h$ is induced by a map $P \to G \to Q$. As in the proof of Theorem 3.5, it follows from the fact that $(X, \mathcal{M}) \to (Y, \mathcal{M}_Y)$ is smooth and exact that the map $P \to Q$ is locally exact. Since the statement is also local on Y_{log}, we may replace $P \to Q$ by the pushout construction as in the proof of 3.5. Thus we may assume that the map $Q^{gp} \to Q^{gp}/P^{gp}$ admits a splitting σ.

We have proper surjective maps of Hausdorff topological spaces

$$X_{log}(\mathcal{M}) \xrightarrow{h_{log}} X_{log}(\mathcal{F}) \xrightarrow{\tau} X.$$

It follows that $X_{log}(\mathcal{F})$ has the quotient topology induced by h_{log}. If $z \in X$, $(\tau \circ h_{log})^{-1}(z)$ is a torsor under $T_{\overline{\mathcal{M}}_z}$ and $\tau^{-1}(z)$ is a torsor under $T_{\overline{\mathcal{F}}_z}$. Hence if $z' \in \tau^{-1}(z)$, $h_{log}^{-1}(z')$ is a torsor under $T_{\overline{\mathcal{M}}_z/\overline{\mathcal{F}}_z}$. Let us identify $X_{log}(\mathcal{M})$ with $A_Q^{log} = T_Q \times X_Q$. Then if $z'' \in (\tau \circ h_{log})^{-1}(z)$ corresponds to $(\zeta, x) \in T_Q \times X_Q$, $x = |z|$ and $F(x)$ is identified with the set of all $q \in Q$ which map to a unit of $\mathcal{M}_z$. Let $G(x)$ be the face of Q generated by $F(x)$ and G. Then $\overline{\mathcal{M}}_z/\overline{\mathcal{F}}_z \cong Q/G(x)$, and the map $T_Q \times X_Q \to X_{log}(\mathcal{F})$ identifies a pair of points (ζ, x) and (ζ', x') if and only if $x = x'$ and $\zeta'\zeta^{-1} \in T_{Q/G(x)} \subseteq T_Q$. Since $G(x)$ contains P, $T_{Q/G(x)}$ can be viewed as a subgroup of $T_{Q/P}$, and $\zeta'\zeta^{-1}$ lies in $T_{Q/G(x)}$ if and only if $T_\sigma(\zeta'\zeta^{-1})$ lies in $T_{Q/G(x)}$ and maps to the identity element of T_P.

Recall the map $\eta'_{A,P}: X_Q \to X_Q(1) \cong X_{Q/P}$ constructed from an effective generating cycle A for C_Q in Corollary 2.12. Consider the following commutative diagram:

$$\begin{tikzcd}
T_Q \times X_Q \arrow[r, "g", "\cong"'] \arrow[d, "h_{log}"'] & (T_{Q/P} \times X_{Q/P}) \times (T_P \times X_P) \arrow[d, "h \times \mu_P"] \\
A^{log}_Q(\mathcal{F}) \arrow[r, dashed] \arrow[dr, "f_{log}(\mathcal{F})"'] & A^{log}_{Q,P} \times A^{log}_P \arrow[d, "pr"] \\
& A^{log}_P .
\end{tikzcd}$$

Here the homeomorphism g comes from the splitting $T_Q \cong T_{Q/P} \times T_P$ induced by σ and the homeomorphism $(\eta'_{A,P}, X_\theta): X_Q \cong X_{Q/P} \times X_P$ of Corollary 2.12. The map $\mu_P: T_P \times X_P \to A^{log}_P$ is the canonical isomorphism (multiplication), h is simply the restriction of h_{log} to the subset $T_{Q/P} \times X_{Q/P}$ of $T_Q \times X_Q$, and $A^{log}_{Q,P}$ is its image in $A^{log}_Q(\mathcal{F})$. Recall from (2) of Corollary 2.12 that for any $x \in X_Q$

$$\langle F(x) + P \rangle = \langle F(\eta'_{A,P}(x)) + P \rangle = F(\eta'_{A,P}(x)),$$

and hence $G(x) = G(\eta'_{A,P}(x))$. Thus g is compatible with the equivalence relations defined by h_{log} and $h \times \mu_P$. It follows that there is a homeomorphism filling in the diagram as shown by the dashed arrow. This proves that $f_{log}(\mathcal{F})$ is a product map, hence a submersion.

It remains only to describe the fiber $A^{log}_{Q,P}$. Note that Q/P is still a fine monoid, but may not be sharp—its group of units is identified with $\langle P \rangle / P$. Let us replace Q by Q/P. The equivalence relation defined by h identifies (ζ, x) and (ζ', x') if and only if $x = x'$ and $\zeta' \zeta^{-1} \in \langle F(x) + G \rangle$. In particular, we have a factorization:

$$\tau_Q: T_Q \times X_Q \xrightarrow{h} A^{log}_{Q,P} \xrightarrow{\tau'} A_Q.$$

Suppose first that $f(\mathcal{F})$ is vertical. This means that $f^*\mathcal{M}_Y$ is not contained in any proper face of $\mathcal{F}$. Thus $\mathcal{F}$ is the sheaf of faces of $\mathcal{M}$ generated by $f^*\mathcal{M}_Y$, and if we have charts P and Q as above, G is the face of Q generated by P. Since now $P = 0$, $G = Q^*$, so $\langle F(x) + G \rangle = F(x)$ for every x. This shows that τ' is an isomorphism. Now condition (b) in the definition of relative smoothness allows us to have chosen $\mathcal{M}$ so that the stalks of $\mathcal{M}/\mathcal{F}$ are free. Then $\overline{Q}$ is a free monoid and Q is the product of $\overline{Q}$ and the group Q^*. It follows that the affine toric analytic space A_Q is smooth, hence a manifold.

If $f(\mathcal{F})$ is not vertical, then G is strictly bigger than Q^*. Choose a finite set of generators $(g_1, \ldots, g_k)$ for the fine submonoid G of Q. Then $G = \langle g_1 + g_2 + \cdots + g_k \rangle$. Consider the map $\mathbf{N} \to Q$ sending 1 to $g_1 + g_2 + \cdots + g_k$. This map is local and locally exact. Furthermore, the corresponding map of log analytic spaces $g\colon (A_Q, \mathcal{F}) \to A_{\mathbf{N}}$ is vertical, and the stalks of the quotient $\mathcal{M}/\mathcal{F}$ are free. As we have just seen, it follows that the map $g^{log}\colon A_Q^{log}(\mathcal{F}) \to A_{\mathbf{N}}^{log}$ is a topological submersion whose fibers are topological manifolds. Since $A_{\mathbf{N}}^{log}$ is a manifold with boundary $B := \tau^{-1}(0)$, it follows that $A_Q^{log}(\mathcal{F}) \cong A_{Q,P}^{log}$ is a topological manifold with boundary $g_{log}^{-1}(B)$. But $g_{log}^{-1}(B)$ is exactly the subset of $A_Q^{log}(\mathcal{F})$ lying over the locus of $A_Q(\mathcal{F})$ where the log structure is not trivial. $\square$

4 Idealized monoids and log spaces

In this section we describe a generalization of our techniques which may prove useful in analyzing strata of toric varieties and log spaces. The only real subtleties are hidden in the meanings of definitions, so we attempt to explain these carefully, only sketching the parts of the arguments that are parallel to the nonidealized version.

By an *idealized monoid* we mean a pair (M, K) where M is a (commutative) monoid and K is an ideal of M. By a *face of* (M, K) we mean a face F of M which does not meet K, or equivalently, such that the corresponding prime ideal contains K. We write $\operatorname{Spec}(M, K)$ for the set of such prime ideals. If $\theta\colon (P, J) \to (Q, K)$ is a homomorphism of idealized monoids, let P^+Q denote the ideal of Q generated by the image of the maximal ideal P^+ of P and let $K_\theta := (P^+Q) \cup K$. Then $\operatorname{Spec}(Q, K_\theta)$ is the set of primes ideals of (Q, K) lying over P^+.

Definition 4.1 *A homomorphism of idealized monoids $\theta\colon (P, J) \to (Q, K)$ is exact if $\theta^{-1}(K) = J$ and for every face P' of (P, J), the restriction θ' of θ to P' is exact.*

Note that every exact homomorphism is local—it sends the closed point of

$\operatorname{Spec}(Q,K)$ (if there is one) to the closed point of $\operatorname{Spec}(P,J)$. Indeed, if $K = Q$ then $\operatorname{Spec}(Q,K)$ is empty and there is nothing to check, so we may assume that K is a proper ideal. Then if $p \in P$ and $\theta(p) \in Q^*$, $\theta(p) \notin K$ so $p \notin J$, and the same is true for every multiple of p. It follows that p does not belong to the radical of J and hence that there is a face P' of (P,J) containing p. Since $P' \to Q$ is exact, it is local, and it follows that p is a unit of P'.

Proposition 4.2 *Let $\theta\colon (P,J) \to (Q,K)$ be a homomorphism of idealized monoids. Consider the conditions:*

1. *$\theta\colon (P,J) \to (Q,K)$ is exact.*
2. *$\operatorname{Spec}\theta\colon \operatorname{Spec}(Q,K) \to \operatorname{Spec}(P,J)$ is surjective.*

Then (1) implies (2), and the converse holds if P and Q are fine and saturated and J and K are radical ideals.

Proof: Suppose (1) holds and F is a face of (P,J). It is easy to verify that $\theta_F\colon (P_F, J_F) \to (Q_F, K_F)$ is still exact and hence is local, as we saw above. Furthermore, since $J = \theta^{-1}(K)$ does not meet F, K_F is a proper ideal of Q_F. Hence Q_F^* is a face of (Q_F, K_F) and its inverse image G in Q is a face of (Q,K) above F. This proves the surjectivity of $\operatorname{Spec}\theta$. Conversely, suppose that (2) holds and that P and Q are fine and saturated. Let F be a face of (P,J), and choose a face G of (Q,K) lying over F. Every face F' of F is a face of (P,J), so there is a face G' of (Q,K) lying over F'. Then $G \cap G'$ is a face of G lying over F'. This shows that $\operatorname{Spec} G \to \operatorname{Spec} F$ is surjective. Since F and G are fine and saturated, it follows that $F \to G$ is exact. Since G is a face of Q, $G \to Q$ is exact, and it follows that $F \to Q$ is also exact. Now since J is a radical ideal, it is the intersection of all the primes $\mathfrak{p}$ of $\operatorname{Spec}(P,J)$, and since each such prime comes from a prime of Q containing K, $J = \theta^{-1}(K)$. $\square$

Definition 4.3 *A morphism of idealized monoids $\theta\colon (P,J) \to (Q,K)$ is*

1. *locally exact if for every face G of (Q,K), the map $(P_F, J_F) \to (Q_G, K_G)$ is exact, where $F = \theta^{-1}(G)$,*
2. *very locally exact if for every face G of (Q, K_θ), the map $(P,J) \to (Q_G, K_G)$ is exact.*

Lemma 4.4 *Let $\theta\colon (P, J) \to (Q, K)$ be a local and very locally exact homomorphism of fine saturated idealized monoids.*

1. *For every face G of $\operatorname{Spec}(Q, K)$, the induced map*

$$F := \theta^{-1}(G) \to G$$

   is locally exact.

2. *θ is locally exact.*

Proof: Let G be a face of (Q, K) and $F := \theta^{-1}(G)$. Let G' be a face of G with $\theta^{-1}(G') = P^*$. Then G' is a face of (Q, K_θ) and by assumption $(P, J) \to (Q_{G'}, K_{G'})$ is exact. Since F is a face of (P, J), the composite $F \to Q_{G'}$ is exact, and hence $F \to G_{G'}$ is also exact. Since this is true for every face G' of G with $\theta^{-1}(G') = P^*$, $F \to G$ is very locally exact. By Theorem 2.3 it is locally exact. This proves (1).

In fact we shall not use (2), but we prove it anyway. It will suffice to replace J and K by their radicals and to work with the associated cones. Again let G be a face of (Q, K) and $F := \theta^{-1}(G)$; we claim that $(P_F, J_F) \to (Q_G, K_G)$ is exact. First we show that if P' is any face of (P, J) containing F, then $P'_F \to Q_G$ is exact. Suppose that $a \in P'^{gp}$ and $\theta(a) \in Q_G$. Then there exists some $g \in G$ with $g + \theta(a) \in Q$. Since we are working with cones, we can apply Theorem 2.3 to the locally exact map $\theta_F\colon F \to G$ induced by θ. Thus we can write $g = \theta(f) + g'$, where g' is contained in a face G' of (G, P^+G) and $f \in F$. Since G is a face of (Q, K), in fact G' is a face of (Q, K_θ), and since θ is very locally exact, $(P, J) \to (Q_{G'}, K_{G'})$ is exact. In particular, $P' \to Q_{G'}$ is exact. Since $\theta(a + f) \in Q_{G'}$, $a + f \in P'$ and hence $a \in P'_F$, as required. Finally, we claim that the inverse image of K_G in P_F is J_F. Indeed, if $\theta(p - f) = k - g$, then we can use the argument above to write $g = g' + \theta(f')$, where $\theta^{-1}\langle g'\rangle = F^*$. Since $(P, J) \to (Q_{G'}, K_{G'})$ is exact, it follows that $p - f \in J_F$. □

Note that the homomorphism:

$$\theta\colon (\mathbf{N}, \emptyset) \to (\mathbf{N} \oplus \mathbf{N}, \mathbf{N}^+ \oplus \mathbf{N}^+) : n \mapsto (n, 0)$$

satisfies condition (1) but it is not very locally exact.

If (Q, K) is an idealized monoid, let

$$C_Q(K) := \bigcup \{C_G : G \text{ is a face of } (Q, K)\},$$

and if $\theta\colon (P, J) \to (Q, K)$ is a homomorphism of idealized monoids, let

$$C_{Q,P}(K) := \bigcup\{C_G : G \text{ is a face of } (Q, K_\theta)\}.$$

Proposition 4.5 *Let $\theta\colon (P, J) \to (Q, K)$ be a local and locally exact homomorphism of fine idealized monoids. Assume that P is sharp. Then the addition map induces a homeomorphism*

$$\sigma\colon C_{Q,P}(K) \times C_P(J) \to C_Q(K).$$

For each pair (F, G), where F is a face of (P, J), and G a face of (Q, K_θ), $\langle F + G\rangle$ is a face of (Q, K), and σ induces a homeomorphism

$$C_{\langle F+G\rangle,F} \times C_G \to C_{\langle F+G\rangle}.$$

Proof: We first check that the addition map sends $C_{Q,P}(K) \times C_P(J)$ to $C_Q(K)$. If $q \in C_{Q,P}(K)$ there exists a face G of (Q, K_θ) containing q, and if $p \in C_P(J)$ there exists a face F of (P, J) containing p. Since $\theta_G\colon (P, J) \to (Q_G, K_G)$ is exact, $\mathrm{Spec}(\theta_G)$ is surjective, so there exists a face G' of (Q, K) containing G such that $\theta^{-1}(G') = F$. Then G' contains $\theta(p) + q$, and hence $\theta(p) + q \in C_Q(K)$, as required. To see that σ is injective, suppose that $(q_i, p_i) \in C_{Q,P}(K) \times C_P(J)$ for $i = 1, 2$, and $q_1 + \theta(p_1) = q_2 + \theta(p_2)$. Let G' be the face of C_Q generated by $\theta(p_i) + q_i$; note that $\theta(p_i)$ and q_i belong to G'. We have seen that G' is a face of (Q, K), so $F' := \theta^{-1}(G')$ is a face of (P, J) and $F' \to G'$ is locally exact, by Lemma 4.4. Then it follows from Theorem 2.3 that $q_1 = q_2$ and $p_1 = p_2$. For the surjectivity, suppose that $q \in C_Q(K)$. Then there is a face G of (Q, K) containing q, and $F := \theta^{-1}(G)$ is a face of (P, J). By Lemma 4.4, $F \to G$ is locally exact, and it follows from Theorem 2.3 that there exist an element q' of G with $\theta^{-1}\langle q'\rangle = \{0\}$ and an element p of F such that $q' + \theta(p) = q$. Then $(q', p) \in C_{Q,P}(K) \times C_P(J)$ and $\sigma(q', p) = q$ as required. Finally, since $C_Q(K)$ admits a locally closed finite cover by the sets C_G as G ranges over the faces of (Q, K) and the restriction of σ^{-1} to each of these is continuous, it follows that σ^{-1} is also continuous. The compatibility expressed in the last statement is clear. □

Now we can prove the analog of the main local rounding result (Theorem 2.5). If (M, K) is an idealized monoid, we write $X_M(K)$ for the set of elements x of X_M which annihilate K. Suppose that $\theta\colon (P, J) \to (Q, K)$ is as in Proposition 4.5, and let

$$\pi := pr \circ \sigma^{-1}\colon C_Q(K) \to C_{Q,P}(K).$$

Choose an effective generating cycle A for C_Q. Then the associated moment map $\mu\colon X_Q \to C_Q$ induces maps $X_Q(K) \to C_Q(K)$ and $X_{Q,P}(K) \to C_{Q,P}(K)$. Let $\nu\colon C_{Q,P}(K) \to X_{Q,P}(K)$ denote the inverse of the latter, and let η be the composite

$$\eta : X_Q(K) \xrightarrow{\mu_A} C_Q(K) \xrightarrow{\pi} C_{Q,P}(K) \xrightarrow{\nu_{A,P}} X_{Q,P}(K).$$

Proposition 4.6 *With the notation and hypotheses above, the map*

$$(\eta, X_\theta)\colon X_Q(K) \to X_{Q,P}(K) \times X_P(J)$$

is a homeomorphism.

Proof: If Q' is a face of (Q,K), let $P' := \theta^{-1}(Q')$ and let $\theta'\colon P' \to Q'$ be the homomorphism induced by θ. Then θ' is locally exact and local, and the restriction of A to Q' is an effective generating cycle for Q'. The map above then restricts to a map

$$(\eta', X_{\theta'})\colon X_{Q'} \to X_{Q',P'} \times X_{P'},$$

which is a homeomorphism by Theorem 2.5. To see that the map is injective, let x_1 and x_2 be two points of $X_Q(K)$ with the same image (y,z) in $X_{Q,P}(K) \times X_P(J)$. Choose faces Q_i of (Q,K) with $x_i \in X_{Q_i}$ and let $P_i := \theta^{-1}(Q_i)$. Since X_{Q_i} is invariant under η, $y \in X_{Q_i}$. It follows that y belongs to $X_{Q_1} \cap X_{Q_2} = X_{Q'}$, where $Q' := Q_1 \cap Q_2$. Similarly, $z \in X_{P'}$, where $P' := P_1 \cap P_2$. Since the map

$$(\eta', X_{\theta'})\colon X_{Q'} \to X_{Q',P'} \times X_{P'}$$

is surjective, there is a point $x' \in X_{Q'}$ mapping to (y,z). Since the restriction of our map to each X_{Q_i} is injective, it follows that $x_1 = x = x_2$. This proves the injectivity of the map. To prove its surjectiviy, let y be a point of $X_{Q,P}(K)$ and z a point of $X_P(J)$, and choose a face G of (Q,K) with $y \in X_G$ and a face F of (P,J) with $z \in X_F$. As we have seen, there is a face Q' of (Q,K) containing $F + G$. By Theorem 2.3, there is a point $X_{Q'}$ mapping to (y,z). This proves the surjectivity. The fact that the bijective map is a homeomorphism follows from the fact that its restriction to a finite closed cover is. $\square$

If (Q,K) is an idealized monoid, we let $A_Q(K)$ denote the closed subspace of A_Q defined by the ideal of $\mathbf{C}[Q]$ generated by K. Then we have a proper surjective map

$$A_Q^{log}(K) := X_Q(K) \times T_Q \to A_Q(K),$$

and we immediately obtain the following analog of Proposition 3.2.

Proposition 4.7 *Let* $\theta\colon (P, J) \to (Q, K)$ *be a locally exact and injective homomorphism of fine idealized monoids. Then the corresponding map*

$$A^{log}_{\theta}\colon A^{log}_{Q}(K) \to A^{log}_{P}(J)$$

is a topological submersion.

An *idealized log analytic space* is a log analytic space X endowed with a sheaf of ideals $\mathcal{K}_X$ in the sheaf of monoids $\mathcal{M}_X$ such that $\alpha_X(k) = 0$ for every local section k of $\mathcal{K}_X$. A morphism $f\colon X \to Y$ of idealized log spaces is required to be compatible with the ideals, so that $f^\flat$ maps $f^{-1}\mathcal{K}_Y$ to $\mathcal{K}_X$. The category of idealized log analytic spaces such that $\mathcal{K}_X$ is empty is equivalent to the usual category of log analytic spaces. An idealized log analytic space is fine if $\mathcal{M}_X$ is fine and the sheaf of ideals $\mathcal{K}_X$ is locally generated by a finite set of sections.

Definition 4.8 *A morphism* $f\colon X \to Y$ *of idealized log analytic spaces is exact if for every* $x \in X$, *the map*

$$f^\flat_x\colon (\mathcal{M}_{Y,f(x)}, \mathcal{K}_{Y,f(x)}) \to (\mathcal{M}_{X,x}, \mathcal{K}_{X,x})$$

is exact .

It follows from Proposition 4.2 that if $\mathcal{M}_X$ and $\mathcal{M}_Y$ are fine and saturated and $\mathcal{K}_X$ and $\mathcal{K}_Y$ are radical ideals, then f is exact if and only if for every $x \in X$, the map

$$\mathrm{Spec}(f^\flat_x)\colon \mathrm{Spec}(\mathcal{M}_{X,x}, \mathcal{K}_{X,x}) \to \mathrm{Spec}(\mathcal{M}_{Y,f(x)}, \mathcal{K}_{Y,f(x)})$$

is surjective.

The following result generalizes Theorem 3.5. We do not give a detailed general treatment of smoothness for morphisms of idealized log spaces here, just taking as a definition the existence of local charts as in the course of the proof below.

Theorem 4.9 *Let* $f\colon X \to Y$ *be a smooth and exact morphism of fine idealized log analytic spaces. Then* $f_{log}\colon X_{log} \to Y_{log}$ *is a topological submersion.*

Proof: We may and shall assume without loss of generality that $\mathcal{K}_X$ and $\mathcal{K}_Y$ are radical ideals. The statement is local, on X and Y, and the smoothness of f then implies that, locally on X and Y, there exist a homomorphism of idealized monoids $\theta\colon (P,J) \to (Q,K)$ and a diagram

$$\begin{tikzcd}
X \arrow[r, "i"] \arrow[rd, "f"'] & Y \times_{A_P(J)} A_Q(K) \arrow[r] \arrow[d, "f'"] & A_Q(K) \arrow[d, "A_\theta"] \\
 & Y \arrow[r, "\beta"] & A_P(J).
\end{tikzcd}$$

Here again the square is Cartesian and i is a strict open immersion, and furthermore the ideals $\mathcal{K}_X$ and $\mathcal{K}_Y$ are generated by K, and J, respectively. If f is exact, one can conclude as in the proof of Theorem 3.5 that θ is very locally exact and hence locally exact. Then the Theorem follows from Proposition 4.6.

□

5 Complements and applications

In this section we give some applications of our results to log geometry. Many of these have already been envisioned, and some also proved. In particular, Sampei Usui proved Theorem 5.1 and its Corollary 5.2 when f is a multi-generalized semi-stable family over a polydisk [17]. Later, in [8], Takeshi Kajiwara and the first author suggested Theorems 3.5 and 5.1 and managed to prove Corollary 5.2 without the use of either of these results (in the coherent case).

Theorem 5.1 *Let $f\colon X \to Y$ be a morphism of log analytic spaces, where Y is fine and X is relatively coherent. Assume that f is proper, separated, exact, and relatively smooth. Then the map $f_{log}\colon X_{log} \to Y_{log}$ is a topological fiber bundle. That is, locally on Y_{log}, it is homeomorphic to a projection mapping $Z \times Y_{log} \to Y_{log}$.*

Proof: Theorem 3.7, tells us that f_{log} is a submersion whose fibers are manifolds with boundary, and L. Siebenmann's [16, Corollary 6.14] asserts that a proper separated topological submersion whose fiber is stratifiable is in fact a fiber bundle. Since a manifold with boundary has a 2-step stratification, we can conclude that f_{log} is a fiber bundle. □

Corollary 5.2 *With the hypotheses of Theorem 5.1, let F be a locally constant abelian sheaf on X_{log}. Then for all integers q, $R^q f_{log*}(F)$ is locally constant on Y_{log}.*

Proof: This follows easily from Theorem 5.1, as explained in [8, Remark B.2.1]. However it may be of some interest to provide a proof which does not depend on Siebenmann's theorem and also does not use a stratification of the fiber. For this we can appeal to the elementary argument outlined in the appendix and Proposition 5.3 below. This argument also shows that the corollary is true even if condition (2b) of Definition 3.6 is not satisfied. (See the remark after Definition 3.6. □

Proposition 5.3 *Let Y be a fine log analytic space. Then Y_{log} is locally triangulizable. In particular, it is locally path connected and semi-locally simply connected.*

Proof: First suppose that $Y = A_P$ with its standard log structure. Then $Y_{log} = X_P \times T_P$. Here X_P and T_P are semialgebraic subsets of $\mathbf{R}^n$, and the map $X_P \times T_P \to A_P$ is algebraic. Since semialgebraic sets are triangulizable [5], the result is certainly true in this case.

Our general statement is local, so we may assume that there is a fine chart $P \to M_Y$ for Y. Such a chart defines a strict morphism of log analytic varieties $Y \to A_P$, and hence a Cartesian diagram

$$\begin{tikzcd}
Y_{log} \arrow[r] \arrow[d] & A_P^{log} \arrow[d] \\
Y \arrow[r] & A_P.
\end{tikzcd}$$

Since the maps $Y \to A_P$ and $A_P^{log} \to A_P$ are analytic maps, the fiber product is semianalytic subset of a suitable affine space, and hence also locally triangulizable by [5]. □

The fiber bundles in Theorem 5.1 can be used to give geometric models of nearby cycles and monodromy. The following result is a first step in this direction; we expect more explicit results could be constructed in many situations.

To motivate the discussion, consider the case of a smooth proper morphism of analytic spaces $f^*: X^* \to D^*$, where D^* is the punctured disk. Then f^* is a fiber bundle, typically nontrivial, and one wants to relate the action of the monodromy group $\pi_1(D^*)$ on the generic fiber to the degeneration of f^*. Suppose that f^* can be extended to a relatively smooth proper map of log analytic spaces $f: X \to D$, where D is the standard log disk. Then the inclusion $D^* \to D_{log}$ is a homotopy equivalence, and the logarithmic model $f_{log}: X_{log} \to D_{log}$ compactifies f^* and remains a fiber bundle. In fact if $P \to D$ is the origin (so that P is a log point), then $P_{log} \cong \mathbf{S}^1$, the inclusion $P_{log} \to D_{log}$ is also a homotopy equivalence, and the restriction of f to the special fiber $X_0 \to P$ determines the bundle $X_0^{log} \to P_{log}$ and hence the monodromy of $X_{log} \to D_{log}$ and of $X^* \to D^*$. Thus it is sensible to restrict attention to smooth maps over log points.

In fact we shall work over a general fine saturated log point P. Then P_{log} is canonically isomorphic to $\mathrm{Hom}(\overline{M}_P, \mathbf{S}^1)$, and its universal cover $\tilde{P}_{log} \to P_{log}$ is given by the exponential map

$$\mathrm{Hom}(\overline{M}_P, \mathbf{R}(1)) \to \mathrm{Hom}(\overline{M}_P, \mathbf{S}^1).$$

Thus the *log inertia group* of P, *i.e.*, the fundamental group $\pi_1(P_{log})$, is $\mathrm{Aut}(\tilde{P}_{log}/P_{log})$ which is canonically identified with $\mathrm{Hom}(\overline{M}_P, \mathbf{Z}(1))$. Now let $f: Y \to P$ be a relatively smooth saturated and exact map of log analytic spaces. Then $f_{log}: Y_{log} \to P_{log}$ is a submersion. Let us consider the Cartesian diagram

$$\begin{tikzcd}
\tilde{Y}_{log} \arrow[r] \arrow[d] & Y \times \tilde{P}_{log} \arrow[d] \\
Y_{log} \arrow[r] & Y \times P_{log}.
\end{tikzcd}$$

The map $\psi: \tilde{Y}_{log} \to Y \times \tilde{P}_{log} \to Y$ is called the *nearby cycle map.* Since f is saturated, the set Y^{st} where f is strict is dense and open in Y, and the map

$$\psi_{st}: Y^{st}_{log} \to Y^{st} \times P_{log}$$

is an isomorphism. Thus the submersion f_{log} is canonically trivialized over the open set Y^{st}_{log}.

Proposition 5.4 *Let $f: Y \to P$ be a relatively smooth, saturated, exact, proper, and separated morphism of log analytic spaces, where P is a log point. Let Z be the fiber of f_{log} over the origin of P_{log}, and let Y' be any compact subset of Y^{st}. Then there exists a trivialization $\tilde{Y}_{log} \cong Z \times \tilde{P}_{log}$ whose restriction to $\tilde{Y}'_{log}$ agrees with the canonical trivialization described above.*

Proof: Identify $\overline{M}^{gp}_P$ with $\mathbf{Z}^r$ and $\tilde{P}_{log}$ with $\mathbf{R}(1)^r$. Choose $n \in \mathbf{N}^+$, let $B := [-2n\pi i, 2n\pi i]^r$, and restrict everything to B. Then the fiber bundle $\tilde{Y}_{log} \to B$ can be trivialized, and we may choose a trivialization $\Phi: \tilde{Y}_{log} \cong Z \times B$. Then Y^{st} becomes an open subset of Z, and the composite

$$\phi: Y^{st} \times B \xrightarrow{\psi_{st}^{-1}} \tilde{Y}_{log} \xrightarrow{\Phi} Z \times B : (y, t) \mapsto (\phi_t(y), t)$$

defines a family of open immersions $\{\phi_t : t \in B\}$, where ϕ_0 is just the inclusion $Y^{st} \subseteq Z$. By Siebenmann's isotopy extension theorem [16, 6.5], there is a homeomorphism $\Phi': Z \times B \to Z \times B$ (over B) whose restriction to $\tilde{Y}'_{log} \times B$ is $\phi_{|Y'}$. Applying III of *op. cit.* we can extend Φ' to sets B with larger and larger n, obtaining the statement of the proposition. $\square$

For each $\theta \in B$, the proposition determines an isomorphism $Z \cong f_{log}^{-1}(e^\theta)$. In particular if $\gamma \in \pi_1(P_{log})$, one obtains isomorphisms $Z \cong Y_1^{log} := f_{log}^{-1}(1)$, which differ by an automorphism T_γ of Y_1^{log}. Note that the restriction of T_γ to the complement of a suitable neighborhood of the nonstrict locus is the identity. Thus the monodromy "lives near the log structure."

Our next goal is to discuss orientation. First we need some preliminary remarks. If X is a fine log analytic space, X_{log} is in general just a topological space. However, over the open set X^* of X where the log structure is trivial, the map $X^*_{log} \to X^*$ is an isomorphism, so X^*_{log} inherits a complex analytic structure. Furthermore, if $X/\mathbf{C}$ is smooth (in the log sense), X^* is dense in X. If $f: X \to Y$ and y is a point of Y_{log}, then the fiber X_y^{log} of f_{log} over y is again just a topological space. However, the map $\tau_X: X_y^{log} \to X_y$ is an isomorphism over the strict locus X_y^{st} of f. Thus $\tau_X^{-1} X_y^{st}$ has a complex analytic structure, but this set need not be dense, even if f is smooth and exact. We will show that in this case there is a larger open subset of X_{log} whose intersection with every fiber is dense and endowed with a natural complex analytic structure.

Definition 5.5 *A morphism $\theta\colon P \to Q$ of integral monoids is small if*

$$\mathrm{Cok}(\theta^{gp})\colon P^{gp} \to Q^{gp}$$

is a torsion group. A morphism of log spaces $f\colon X \to Y$ is small if for every $x \in X$, the morphism $\overline{M}^{gp}_{Y,f(x)} \to \overline{M}^{gp}_{X,x}$ is small.

Lemma 5.6 *Let $\theta\colon P \to Q$ be a morphism of fine monoids.*

1. *If $\overline{\theta}$ is small, then θ is vertical.*
2. *If $\overline{\theta}$ is exact and small, then for every $q \in Q$, there exist a positive integer n, a $u \in Q^*$, and a $p \in P$ such that $nq = u + \theta(p)$.*

Proof: There is an exact sequence:

$$\mathrm{Cok}(\theta^*) \to \mathrm{Cok}(\theta^{gp}) \to \mathrm{Cok}(\overline{\theta}^{gp}) \to 0.$$

If $\overline{\theta}$ is small, $\mathrm{Cok}(\overline{\theta}^{gp})$ is a finite group, and hence the image $\overline{Q}/\overline{P}$ of $\overline{Q}$ in $\mathrm{Cok}(\overline{\theta}^{gp})$ is a finite integral monoid, hence a group. It follows that Q/P is also a group, so that θ is vertical. If $\overline{\theta}$ is exact, so is θ. For any $q \in Q$, there exist a positive integer n, a unit u of Q, and elements p_1, p_2 of P such that $nq + u = \theta(p_2) - \theta(p_1)$. Since θ is exact, it follows that there is some $p \in P$ such that $\theta(p_2) - \theta(p_1) = \theta(p)$. $\square$

Proposition 5.7 *Let $P \to Q$ be a local and locally exact homomorphism of fine monoids, with P sharp. Let G be a face of Q such that $G^{gp} \cap P^{gp} = \{0\}$. Then the following are equivalent.*

1. *G is maximal among all faces of Q such that $G^{gp} \cap P^{gp} = \{0\}$.*
2. *$P \to Q/G$ is small.*
3. *The natural map $C^{gp}_G \oplus C^{gp}_P \to C^{gp}_Q$ is an isomorphism.*

We shall call a face G satisfying the above conditions cosmall.

Proof: Let us first note that when $P \to Q$ is injective and locally exact and G is a face of Q, $G^{gp} \cap P^{gp} = 0$ if and only if $G \cap P = 0$. Since $P \to Q$ is locally exact, so is the map $P \to Q/G$, and since $G^{gp} \cap P^{gp} = \{0\}$, the map $P \to Q' := Q/G$ is still injective. Corollary 2.4 applies to $C_P \to C_{Q'}$, and we can conclude that the summation map $C_{Q',P} \times C_P \to C_{Q'}$ is bijective. Now suppose that (1) holds, let q be an element of Q, and let q' be its image in Q'. Then there exist a face G' of Q' such that $G'^{gp} \cap P^{gp} = \{0\}$ and elements $g' \in C_{G'}$ and $p \in C_P$ such that $q' = p + g'$. The inverse image of G' in Q is a face G'' of Q such that $G''^{gp} \cap P^{gp} = \{0\}$, and hence $G'' = G$. Thus $g' = 0$. This shows that $C_P \to C_{Q/G}$ is surjective, and hence that $P \to Q/G$ is small. Thus (1) implies condition (2), which is clearly equivalent to (3). Furthermore, if $P \to Q/G$ is small and injective, then the faces of Q containing G correspond bijectively to the faces of P, and in particular G is maximal among those faces of Q which meet P in $\{0\}$. □

Proposition 5.8 *Let $f\colon X \to Y$ be an exact and smooth morphism of fine log analytic spaces.*

1. *The set X^{sm} of points x in X where f is small is open and dense in every fiber.*

2. *Suppose f is small. Then for each point y of Y, the reduced fiber $X_{y_{red}}$ is smooth, and the natural map $X^{log}_y \to X_{y_{red}}$ is a finite covering space. In particular, there is a unique complex analytic structure on X^{log}_y such that this map is complex analytic.*

Proof: Statement (1) is local on X, so by the argument at the beginning of the proof of Theorem 3.5, we may assume that f is given by a local and locally exact morphism of fine monoids $\theta\colon P \to Q$, where P is sharp. We view θ as an inclusion. The fiber of $\operatorname{Spec} Q \to \operatorname{Spec} P$ over P^+ is the closed subset defined by the ideal J of Q generated by P^+, and consists of those primes $\mathfrak{p}$ of Q such that $\mathfrak{p} \cap P = P^+$. These are the primes corresponding to the faces G of Q such that $G \cap P = \{0\}$. Note that since $P \to Q/G$ is exact, it follows that $G^{gp} \cap P^{gp} = \{0\}$. By Proposition 5.7, the cosmall faces of Q correspond exactly to the minimal primes of $Z(J)$. Thus the set of all such primes is dense and open in $Z(J)$. Statement (1) follows.

Replace Q by Q_G, so that $P \to \overline{Q}$ is small. Then if $q \in Q^+$, there is an $n \in \mathbf{Z}^+$ such that $nq \in J$. It follows that the radical of J is Q^+, and hence that the reduced fiber of $A_Q \to A_P$ is A_{Q^*}, which is smooth. Furthermore, the splitting $C_Q \cong C_{\overline{Q}} \times C_{Q^*}$ makes it clear that $X_{Q,P} \cong X_{Q^*}$. Moreover, $Q^* \cap P^{gp} = \{0\}$, so we have exact sequences:

$$0 \to Q^* \to Q^{gp}/P^{gp} \to \overline{Q}^{gp}/P^{gp} \to 0$$

$$0 \to T_{\overline{Q}/P} \to T_{Q/P} \to T_{Q^*} \to 0,$$

and the map $T_{Q/P} \to T_{Q^*}$ is a covering space. Thus the fiber of $A_Q^{log} \to A_P^{log}$ is $X_{Q^*} \times T_{Q/P}$, which maps to A_{Q^*} (the reduced fiber) in the evident way. This proves (2), at least locally on X. But we already know that $\tau\colon X_{log} \to X$ is proper, and it follows that $X_y^{log} \to X_{y_{red}}$ is a covering space. □

Remark 5.9 Suppose in the situation of Proposition 5.8 that the log structure $\mathcal{F} \to \mathcal{O}_X$ is only relatively coherent and that f is exact and relatively smooth. Then we can find an open subset U of X which is dense in every fiber and such that $\mathcal{F}_{|U}$ is coherent and $U \to Y$is small. To prove this we may replace X by an open subset on which $\mathcal{F} \to \mathcal{O}_X$ is relatively coherent in a log structure $\mathcal{M} \to \mathcal{O}_X$. Furthermore we may assume that $X(\mathcal{M}) \to Y$ admits a chart subordinate to an exact and locally exact morphism $P \to Q$ as in the proof of Theorem 3.7. Then the small locus of $X(\mathcal{M}) \to Y$ satisfies the stated conditions. Indeed, this locus is open and dense in every fiber. Furthermore, if $X(\mathcal{M}) \to Y$ is small at a point x of X, then it is vertical, so $\mathcal{F}_x = \mathcal{M}_x$. The set U may not be unique, so we call it *a coherent small locus* of f.

We shall now discuss Poincaré-Verdier duality. We assume from now on that all our spaces are Hausdorff. Let $f\colon X \to Y$ be an exact and relatively smooth morphism of log analytic spaces, where Y is fine and X is relatively coherent. It follows from Theorem 3.7 that the fiber dimension of $f_{log}\colon X_{log} \to Y_{log}$ is a locally constant function on X_{log}, whose values are always even. Let us denote this dimension by $2d_{X/Y}$.

Theorem 5.10 *Let $f\colon X \to Y$ be an exact and relatively smooth morphism of log analytic spaces, where Y is fine and X is relatively coherent. Let $j\colon X^v \to X$ be the inclusion of the vertical locus of f and let $f^v := f \circ j$.*

1. *The functor*

$$Rf_{log!}\colon D^+(X_{log}) \to D^+(Y_{log})$$

admits a right adjoint $Rf^!_{log}$, and for any G,

$$Rf^!_{log}G \cong f^{-1}_{log}G \otimes Rf^!_{log}(\mathbf{Z}).$$

2. *There is a canonical isomorphism*

$$Rf^!_{log}(\mathbf{Z}) \cong j^{log}_!(\mathbf{Z}_{X^v_{log}}[2d_{X/Y}])$$

uniquely determined by the fact that its restriction to each of the fibers of a coherent small locus of f_{log} *(see Remark 5.9) is the canonical one described in Proposition 5.8.*

3. *Let* F *be an object of* $D^+(X_{log})$ *and* G *be an object of* $D^+(Y_{log})$*. Then there is a natural isomorphism:*

$$R\mathcal{H}om(Rf_{log!}F, G) \cong Rf_{log*}R\mathcal{H}om(F, j_!^{log} f_{log}^{v\,-1} G[2d_{X/Y}])$$

In particular, if f *is proper and* $d_{X/Y}$ *is a constant* d*,*

$$\mathcal{H}om(R^q f_{log*}(\mathbf{Q}), \mathbf{Q}) \cong R^{2d-q} f^v_{log!}(\mathbf{Q}).$$

Proof: The existence of the adjoint $Rf_{log_!}$ follows from the fact that $Rf_{log!}$ has finite cohomological dimension, [9, 3.1.5], which in turn follows from the fact that the fibers of f_{log} are manifolds with boundary [7, III§9]. The formula for $Rf^!_{log}$ follows from the fact that f is a topological submersion; see [9, 3.3.2] and [19, 5.1].

The main difficulty is (2). We begin with the following lemma.

Lemma 5.11 *Consider a Cartesian square*

$$\begin{tikzcd}
X' \arrow[r, "j"] \arrow[d, "f'"'] & X \arrow[d, "f"] \\
Y' \arrow[r, "i"] & Y
\end{tikzcd}$$

of locally compact Hausdorff spaces. Assume that $Rf_!$ *has finite cohomological dimension, so that* $Rf^!$ *exists [9, 3.1.5].*

1. *Then* $Rf'_!$ *has finite cohomological dimension, and there is a natural map:* $j^{-1} \circ Rf^! \to Rf'^! \circ i^{-1}$.

2. *This map is an isomorphism if* f *is a topological submersion and* Y *and* Y' *are locally connected.*

3. *If* Y *is locally connected and* f *is a topological submersion and whose fibers are manifolds of dimension* n*, then* $R^! f(\mathbf{Z}_Y)$ *is locally isomorphic to* $\mathbf{Z}_X[n]$*, and its formation commutes with base change.*

Proof: Statement (1) is [9, 3.1.9]. Statement (2) is a local problem so we may assume that f is a projection mapping $Y \times Z \to Y$. Then the square in the lemma becomes identified with the left square of the following diagram:

$$\begin{tikzcd}
Y' \times Z \arrow[r, "j"] \arrow[d, "f'"'] & Y \times Z \arrow[r, "q"] \arrow[d, "f"] & Z \arrow[d, "g"] \\
Y' \arrow[r, "i"] & Y \arrow[r, "p"] & pt.
\end{tikzcd}$$

Let $q' := q \circ j$ and $p' := p \circ i$. Then there is a commutative diagram

$$\begin{tikzcd}
j^{-1}q^{-1}(Rg^!\mathbf{Z}) \arrow[r, "j^{-1}(a)"] \arrow[d, "\cong"'] & j^{-1}(Rf^!\mathbf{Z}) \arrow[d, "b"] \\
q'^{-1}Rg^!(\mathbf{Z}) \arrow[r, "a'"] & Rf'^!(\mathbf{Z})
\end{tikzcd}$$

in which a is the base change map for the square on the right, a' is the base change map for the outer rectangle, and b is the base change map for the square on the left. Thus we are reduced to proving that a and a' are isomorphisms. If suffices to treat a, corresponding to the case of the square on the right. This statement (under a different hypothesis) is asserted without proof in [19, §5]. To prove it in our situation, let $K^{\cdot}_Z$ be the Godement resolution of the constant sheaf $\mathbf{Z}_Z$ on Z. This is a complex of Z-soft and flat sheaves, and hence for any open set V of Z, $Rg^!(\mathbf{Z})(V) = \mathcal{H}om(g_!(K^{\cdot}_V), \mathbf{Z})$. On the other hand, $q^{-1}(K^{\cdot}_Z)$ is an f-soft and flat resolution of $\mathbf{Z}_{Y \times Z}$, and hence if U is an open set in Y, $Rf^!(\mathbf{Z})(U \times V) = \mathcal{H}om(f_!(q^{-1}K^{\cdot}_Z)_{U \times V}, \mathbf{Z})$. If $V = Z$ and $U = Y$, the proper base change theorem says that the natural map $p^{-1}g_!(K^{\cdot}_Z) \to f_!(q^{-1}K^{\cdot}_Z)$ is an isomorphism. Here $K^{\cdot} := g_!(K^{\cdot}_Z)$ is a complex of abelian groups, and so $p^{-1}g_!(K^{\cdot}_Z)$ is a constant complex of sheaves on Y. If U is any connected open subset of Y, $p^{-1}g_!(K^{\cdot}_Z)(U) = K^{\cdot}$, and since Y is locally connected, it follows that, if Y is also connected,

$$\mathrm{Hom}_Y(p^{-1}g_!(K^{\cdot}_Z), \mathbf{Z}) = \mathrm{Hom}(K^{\cdot}, \mathbf{Z}) = \Gamma(Z, Rg^!(\mathbf{Z})).$$

In other words, the natural map

$$\Gamma(Z, Rg^!(\mathbf{Z})) \to \Gamma(Y \times Z, Rf^!(\mathbf{Z}))$$

is an isomorphism if Y is connected. Now if $z \in Z$ and $y \in Y$ and we take the limit over all connected neighborhoods of y and all neighborhoods of z, we see that the stalk of the base change map $q^{-1}Rg^{!}(\mathbf{Z}) \to Rf^{!}(\mathbf{Z})$ is an isomorphism, as required. Statement (3) follows from (2) and the fact that $Rf^{!}(\mathbf{Z}) = \mathbf{Z}[n]$ when f is the projection from $\mathbf{R}^n$ to a point. □

In the situation of (3) of the Lemma, the sheaf $o(f) := \mathcal{H}^{-n}(Rf^{!}(\mathbf{Z}))$ is called the *relative orientation sheaf* of the morphism f. In particular, if f is the projection from a smooth manifold M to a point, $o(f)$ is the usual orientation sheaf of M. By part (2) of the above lemma, if f is a topological submersion whose fibers are manifolds, then, at least over a locally connected base, the restriction of $o(f)$ to a fiber is just the orientation sheaf of the fiber.

Corollary 5.12 *Let $f\colon X \to Y$ be a smooth morphism of complex analytic spaces, of relative dimension d. Then the relative orientation sheaf $o(f)$ is constant, and admits a unique trivialization whose restriction to the fibers is the canonical orientation coming from its structure as a complex manifold.*

Let us return to the situation of the theorem. Our goal is to prove that $o(f_{log})$ is constant. Let us first consider the case when f is given by a morphism of monoids $\theta\colon P \to Q$.

Lemma 5.13 *Let $\theta\colon P \to Q$ be a homomorphism of fine saturated monoids. Assume that P is sharp and Q is torsion free and that θ is local and locally exact. Let $\mathcal{M}_Q$ denote the usual log structure on A_Q, let $\mathcal{F} \subseteq \mathcal{M}_Q$ denote the sheaf of faces generated by P, and assume that the stalks of $\mathcal{M}_Q/\mathcal{F}$ are free. Then the relative orientation sheaf $o(A^{log}_{\theta}(\mathcal{F}))$ is constant. In fact there is a unique global generator of $o(A^{log}_{\theta}(\mathcal{F}))$ whose restriction to each of the fibers of a small locus of $A^{log}_{Q}(\mathcal{F}) \to A^{log}_{P}$ induces the orientation coming from the complex analytic structure constructed in Proposition 5.8.*

Proof: Choose a positive integer n, and consider the diagram

$$\begin{tikzcd}
Q \arrow[r, "\eta"] & \tilde{Q} \\
P \arrow[u, "\theta"] \arrow[r, "n_P"] & P \arrow[u, "\tilde{\theta}"']
\end{tikzcd}$$

where $\tilde{Q}$ is the pushout in the category of saturated monoids. By Theorem (A.3.4) and Tsuji's theorem (A.4.2) of [6], there exists a choice of n such that the morphism $\tilde{\theta}$ is saturated. It follows then that the map $P^{gp} \to \tilde{Q}^{gp}$ admits a section, [6, A.4.1].

Let $X(\mathcal{M})$ (resp. $X(\mathcal{F})$) denote A_Q with the log structure coming from $Q \to \mathbf{C}[Q]$ (resp. $\mathcal{F} \to \mathcal{O}_{A_Q}$), and let Y be A_P with the log structure coming from $P \to A_P$. Let $f\colon X(\mathcal{F}) \to Y$ (resp. $\tilde{f}\colon \tilde{X}(\mathcal{F}) \to Y$) be the map induced by A_θ (resp. $A_{\tilde{\theta}}$). Let X^{sm} denote the open subset of X where $X(\mathcal{M}) \to Y$ is small. Recall from Remark 5.9 that $X^{sm}(\mathcal{F}) \to Y$ is also small and that $\mathcal{F} = \mathcal{M}$ on X^{sm}. Then we have the following Cartesian diagrams:

$$\begin{tikzcd}
\tilde{X}_{log}(\mathcal{F}) \arrow[r, "\eta"] \arrow[d, "\tilde{f}_{log}"'] & X_{log}(\mathcal{F}) \arrow[d, "f_{log}"] \\
Y_{log} \arrow[r, "n"] & Y_{log}
\end{tikzcd}
\qquad
\begin{tikzcd}
\tilde{X}^{sm}_{log}(\mathcal{F}) \arrow[r, "\eta"] \arrow[d, "\tilde{f}^{sm}_{log}"'] & X^{sm}_{log}(\mathcal{F}) \arrow[d, "f^{sm}_{log}"] \\
Y_{log} \arrow[r, "n"] & Y_{log}.
\end{tikzcd}$$

Since $\tilde{f}^{sm}$ is saturated and small, it is strict [6, A.4.1], and it follows that the diagram

$$\begin{tikzcd}
\tilde{X}^{sm}_{log}(\mathcal{F}) \arrow[r] \arrow[d, "\tilde{f}^{sm}_{log}"'] & \tilde{X}^{sm}(\mathcal{F}) \arrow[d, "\tilde{f}^{sm}"] \\
Y_{log} \arrow[r] & Y
\end{tikzcd}$$

is Cartesian. Since $\tilde{f}^{sm}$ is strict and smooth, it is a smooth map of complex analytic varieties without the log structures. By Corollary 5.12, its relative orientation sheaf $o(\tilde{f}^{sm})$ is constant and admits a unique trivialization $\tilde{\gamma}^{sm}$ compatible with the orientations of the fibers coming from their complex analytic structures. By base change, $\tilde{\gamma}^{sm}$ induces a trivialization of $o(\tilde{f}^{sm}_{log})$ with the same property and which we denote by the same symbol.

Since $\tilde{Q}^{gp} \cong P^{gp} \oplus (Q/P)^{gp}$, it follows that $T_{\tilde{Q}} \cong T_P \times T_{Q/P}$. Then as we saw in the proof of Theorem 3.7, the choice of an effective generating cycle for Q defines a homeomorphism $\tilde{X}_{log}(\mathcal{F}) \cong Y_{log} \times A^{log}_{Q,P}$ sending $\tilde{f}_{log}$ to the projection mapping; we also saw that $A^{log}_{Q,P} \cong A_{Q/P}$ and is a complex manifold. Then it follows from Lemma 5.11 that $o(\tilde{f}_{log})$ is constant. Since X^* is connected and dense in $X^{sm}_{log}(\mathcal{F})$, $\tilde{X}^{sm}_{log}(\mathcal{F})$ is connected, and since $\tilde{X}^*$ is also dense in $\tilde{X}_{log}(\mathcal{F})$, there is a unique extension $\tilde{\gamma}$ of $\tilde{\gamma}^{sm}$ to $\tilde{X}_{log}(\mathcal{F})$.

We claim that $\tilde{\gamma}$ descends to a trivialization γ of $o(f_{log})$. The restriction of $\tilde{\gamma}$ to $\tilde{X}^*$ descends to X^*, since $f^*\colon X^* \to Y^*$ is a smooth map of smooth complex manifolds. It follows that the two pullbacks of $\tilde{\gamma}$ to $\tilde{X}_{log}(\mathcal{F}) \times_{X_{log}(\mathcal{F})} \tilde{X}_{log}(\mathcal{F})$ agree on $\tilde{X}^* \times_{X^*} \tilde{X}^*$, and since this set is dense, they agree everywhere. Since η is a finite covering space, this implies that $\tilde{\gamma}$ descends to a trivialization γ of $o(f_{log})$ on $X_{log}(\mathcal{F})$, as required. Since $\tilde{\gamma}^{sm}$ is compatible with the complex analytic structure of the fibers of f^{sm}_{log}, the same is true of γ^{sm}. □

Now we prove (2) of Theorem 5.10 with the assumption that f is vertical. Since in this case f_{log} is a submersion whose fibers are manifolds, $o(f_{log})$ is defined and locally isomorphic to $\mathbf{Z}$. Locally on X, f admits a chart, and hence by Lemma 5.13, X admits an open covering on which there exist isomorphisms as in (2). Since the small locus is dense in every fiber, these local isomorphisms agree, and hence patch to a unique isomorphism on all of X_{log}.

For the general case, let X^v be the vertical locus of f, and recall from Theorem 3.7 that the pair (X_{log}, X^v_{log}) is locally isomorphic to $Y_{log} \times (M, M^*)$ where M is a manifold with boundary $M \setminus M^*$. Then the vertical case gives an isomorphism $\mathbf{Z}[2d_{X/Y}] \cong j^{-1}_{log} Rf^!_{log}(\mathbf{Z})$ and hence a map

$$j_{log!}\mathbf{Z}[2d_{X/Y}] \to Rf^!_{log}(\mathbf{Z}).$$

Since both sides commute with base change, it is enough to check that this map is an isomorphism along the fibers. Then the result follows from the standard computation of the dualizing complex of a manifold with boundary [7, V, Example 2.9].

Now we apply Verdier duality [9, 3.1.10] to obtain

$$\begin{aligned}
R\mathcal{H}om(Rf_{log!}F, G) &\cong Rf_{log*}R\mathcal{H}om(F, Rf^!_{log}G) \\
&\cong Rf_{log*}R\mathcal{H}om(F, f^{-1}_{log}G \otimes Rf^!_{log}\mathbf{Z}) \\
&\cong Rf_{log*}R\mathcal{H}om(F, f^{-1}_{log}G \otimes j^{log}_!\mathbf{Z}[2d_{X/Y}]) \\
&\cong Rf_{log*}R\mathcal{H}om(F, j^{log}_! j^{-1}_{log} f^{-1}_{log} G[2d_{X/Y}]).
\end{aligned}$$

□

Corollary 5.14 *With the hypotheses of Theorem 5.10, the fibers of* f_{log} *are orientable manifolds with boundary.*

6 Appendix: Local constancy of sheaves

The purpose of this appendix is to give an elementary proof, based on the introductory discussion of vanishing cycles in [4, Exposé 1], of the following basic result.

Theorem 6.1 *Let* $f\colon X \to S$ *be a proper separated submersion and let* F *be a locally constant sheaf of abelian groups on* X*. Assume that* S *is locally path connected and semi-locally simply connected [2, II §6]. Then for each* $q \in \mathbf{Z}$*,* $R^q f_* F$ *is locally constant on* S*.*

Proof: First we shall prove this when S is the unit interval. This case relies on the following criterion.

Proposition 6.2 *Let* F *be a sheaf of abelian groups on the unit interval* I*. For* s, t *with* $0 \leq s < t \leq 1$*, let* $J := [s,t]$*, let* F_J *be the restriction of* F *to* J*, and let* $j : [s,t) \to [s,t]$ *and* $j' : (s,t] \to [s,t]$ *be the inclusions. Suppose that for every such* s *and* t*, the maps*

$$\begin{aligned} F_J &\to j_* j^* F_J \text{ and} \\ F_J &\to j'_* j'^* F_J \end{aligned}$$

are isomorphisms. Then F *is constant.*

Proof: Suppose that the first condition always holds. We claim that for every $t' > t > s$, the map $F([s,t')) \to F([s,t))$ is injective. Suppose f is a section of $F([s,t'))$ which vanishes on $[s,t)$. Let T be the set of all t'' such that f vanishes on $[s,t'')$. Evidently $t \in T$. Let b be the supremum of T. For every $b' < b$, there is some $t'' \in T$ such that $t'' > b'$. Then f vanishes on $[s,t'')$ and hence also on $[s,b')$. Since the set of all $[s,b')$ with $b' < b$ covers $[s,b)$, it follows that $b \in T$. Say $b < t'$. Let $J := [s,b]$, and observe that since $F_J \to j_* j^* F_J$ is an isomorphism, f also vanishes on $[s,b]$, and hence in some neighborhood of b. This contradicts the fact that b is an upper bound for T. Hence $b = t'$, proving the injectivity.

Next we claim that for every $t \geq s$ the map $F([s,1]) \to F([s,t])$ is surjective. Suppose f is a section of F over $[s,t]$. We may assume $t < 1$.

Hence there exist some $t' > t$ and $f' \in F([s, t'))$ such that f' extends f. We know f' is unique by the injectivity proved above. Let T be the set of all t' for which there exists such an f' on $[s, t')$ and let b be the supremum of T. Then for each $b' < b$, there is some $t' > b'$ in T. Hence there is a (unique) f' on $[s, t')$ extending f. The set of all $[s, t')$ covers $[s, b)$ and by the uniqueness these f' patch to an extension of f to $[s, b)$. Thus $b \in T$. By assumption, f' extends to $[s, b]$. If $b = 1$ we are done, and otherwise we find a contradiction.

It follows that for every $s < t$, the map $F([s, 1]) \to F([s, t])$ is bijective, and in the same way one proves that the map $F([0, 1]) \to F([s, 1])$ is bijective. Then it follows that F is constant. □

Lemma 6.3 *Let $f\colon X \to I$ be a topological submersion, and let F be a locally constant abelian sheaf on X. Let $X' := f^{-1}[0, 1)$ and let $k\colon X' \to X$ be the inclusion. Then the natural map $F \to Rk_*k^*F$ is an isomorphism in $D^+(X)$. Hence for every $q \in \mathbf{Z}$, the natural map*

$$R^q f_* F \to R^q (f \circ k)_* (k^* F)$$

is an isomorphism.

Proof: Let Z be the complement of X' in X. It suffices to show that the cohomology sheaves $\mathcal{H}^q_Z(X, F)$ are all zero. This is a local condition on X, so we may assume that F is constant and that $X = Y \times J$, where $J := (r, 1]$ for some $r < 1$. We claim that the map

$$H^q(Y \times J, F) \to H^q(Y \times J', F)$$

is an isomorphism, where $J' := (r, 1)$. Choose $r' \in J$. Then the map sending J (resp. J') to r', followed by the inclusion of $\{r'\}$ in J (resp. J') is homotopic to the identity. It follows that the inclusion of $Y \times J'$ in $Y \times J$ is a homotopy isomorphism. Hence it induces an isomorphism on cohomology, by [9, 2.7.5]. □

Lemma 6.4 *Theorem 6.1 is true if $S = I$.*

Proof: We prove by induction on q that $\mathcal{F}^q := R^q f_* F$ is constant on I. Assume this is true for $q' < q$. Let $j\colon I' := [0, 1) \to I$ be the inclusion and let

$f' \colon X' \to I'$ be the map induced by f. We have a commutative diagram

$$\begin{tikzcd}
\mathcal{F}^q = R^q f_* F \arrow[r] \arrow[d] & R^q(f \circ k)_* k^* F \cong R^q(j \circ f')_* k^* F \arrow[d] \\
j_* j^* R^q f_* F \arrow[r] & j_* R^q f'_* k^* F.
\end{tikzcd}$$

The top vertical arrow is an isomorphism by the lemma. The horizontal arrow on the bottom is trivially an isomorphism. The vertical arrow on the right is the edge homomorphism associated with the spectral sequence with $E_2^{p,q'} = R^p j_* R^{q'} f'_* F$. The induction hypothesis implies that $R^{q'} f'_* F$ is constant on I' if $q' < q$ and hence that $E_2^{p,q'} = 0$ for $q' < q$ and $p > 0$. This implies that the right vertical arrow is an isomorphism. It follows that the left vertical arrow is also an isomorphism.

Now if $0 \leq s < t \leq 1$, let $J := [s,t]$ and let $i \colon J \to I$ be the inclusion. Let $f_J \colon X_J \to J$ be the pullback of f to J and let F_J be the pullback of F to X_J. By the proper base change theorem, the natural map $i^* \mathcal{F}^q \to R^q f_{J*} F_J$ is an isomorphism for all q. The above argument applies equally to f_J, and it follows that the natural map $\mathcal{F}^q_J := i^* \mathcal{F}^q \to j_* j^* (\mathcal{F}^q_J)$ is an isomorphism, as in Proposition 6.2. The statement for j' is proved similarly. It follows that $\mathcal{F}^q$ is constant. $\square$

Lemma 6.5 *Let S be a locally path connected and semi-locally simply connected space, and let $\mathcal{F}$ be an abelian sheaf on S. Assume that for every path $\gamma \colon I \to S$, the pullback $\gamma^* \mathcal{F}$ is constant. Then $\mathcal{F}$ is locally constant.*

Proof: First we prove this when $S = I \times I$. We show first that if U is a path connected subset of S and $x \in U$, the map $\mathcal{F}(U) \to \mathcal{F}_x$ is injective. Indeed, if f is an element of the kernel and $y \in U$, there is a path γ in U from x to y, and $\gamma^* \mathcal{F}$ is constant. It follows that the stalk of f at y also vanishes, and since this is true for every y, f vanishes.

Next we prove that the map $\mathcal{F}(S) \to \mathcal{F}_x$ is bijective, where $x := (0,0)$. Choose $g \in \mathcal{F}_x$. Since the restriction of $\mathcal{F}$ to $I \times 0$ is constant, g extends uniquely to a section of $\mathcal{F}$ on $I \times 0$. Let T be the set of all t such that g extends to a section of $\mathcal{F}$ on $I \times [0,t]$, and let b be the supremum of T. We claim that $b \in T$. If $b = 0$ this is certainly true. If $b > 0$, then for every $b' < b$, there is a unique section of $\mathcal{F}$ on $I \times [0,b')$ extending g, and these patch to a section f on $I \times [0,b)$. For each $a \in I$, the restriction of $\mathcal{F}$ to $a \times [0,b]$ is constant, so there is a unique section f_a of $\mathcal{F}$ on $a \times [0,b]$ which agrees with f on $a \times [0,b)$. Then there exist an open box U around (a,b) and a section f_U of $\mathcal{F}$ on U whose stalk at (a,b) agrees with the stalk of f_a at (a,b). That is, for some $b' < b$, f_U and f_a agree on $a \times (b',b]$. Then $V := U \cap I \times [0,b)$ is path connected, and the stalks of f_U and f agree at the points of V on $a \times (b',b)$. By the injectivity proved above, this implies that they agree on all of V and hence patch to a global section on $I \times [0,b) \cup U$. Doing this for each a, we find a section of $\mathcal{F}$ on $I \times [0,b]$ whose stalk at x is g. This shows that $b \in T$. If $b < 1$ we see from the compactness of $I \times [0,b]$ that f extends to an open neighborhood of $I \times [0,b]$ in $I \times I$ [9, 2.5.2]. But this would contradict the fact that b is an upper bound for T. Thus $b = 1$, and we are done.

Now suppose that S is connected, simply connected and locally path connected. Given any two points x and y in S, there is a path γ from x to y, and hence a map $\phi_{y,x}$ from $\mathcal{F}_x \to \mathcal{F}_y$. Since S is simply connected, any two paths are homotopic, and by the preceding paragraph, this isomorphism is independent of the choice of path. It follows that if z is a third point of S, $\phi_{z,y}\phi_{y,x} = \phi_{z,x}$. Note that if f is a section of $\mathcal{F}$ over some connected open subset U of S and x and y belong to U, then $\phi_{y,x}(f_x) = f_y$.

Choose some $s \in S$. We claim that for each open subset U of S and each $f \in \mathcal{F}_s$, there is a unique $g \in \mathcal{F}(U)$ such that $g_u = \phi_{u,s}(f)$ for all $u \in U$. The uniqueness is clear. For the existence, consider pairs (U', g') where U' is an open subset of U and $g' \in \mathcal{F}(U')$ is such that $g'_{u'} = \phi_{u',s}(f)$ for every $u' \in U'$. Order the set of such pairs as usual, and suppose that (V, h) is maximal (by the sheaf axiom such a pair exists). Say $u \in U$, and choose a connected open neighborhood U' of u and a $g' \in \mathcal{F}(U')$ such that $g'_u = \phi_{u,s}(f)$. If u' is any point of U', it follows that $g'_{u'} = \phi_{u',u}(g'_u) = \phi_{u',u}\phi_{u,s}(f) = \phi_{u',s}(f)$. In particular, if $u' \in U' \cap V$, $g'_{u'} = h_{u'}$. This implies that the restriction of g' to $U' \cap V$ agrees with the restriction of h to $U' \cap V$. Then h extends to $U' \cup V$, so by maximality $U' \subseteq V$. Hence $V = U$. Thus we have constructed, for every open U in S, a map $\sigma_U \colon \mathcal{F}_s \to \mathcal{F}(U)$ such that $\sigma_U(f)_u = \phi_{u,s}(f)$ for all $u \in U$. It is clear that the maps $\mathcal{F}_s \to \mathcal{F}(U)$ are compatible with restriction, and hence define a map from the constant sheaf $\mathcal{F}_s$ to the sheaf $\mathcal{F}$. It follows from our assumption that this map is an isomorphism on stalks. Hence it is an isomorphism and $\mathcal{F}$ is constant.

Now we prove the general case. Since S is locally path connected, its connected components are open and locally path connected, and it is enough to prove the result for each connected component. Since S is semi-locally simply connected, it has a universal cover $\tilde{S} \to S$. By the previous case, the pullback of $\mathcal{F}$ to $\tilde{S}$ is constant. Since $\tilde{S} \to S$ is a covering space, it follows that $\mathcal{F}$ is locally constant. $\square$

Now we can finish the proof of Theorem 6.1. Let $\gamma \colon I \to S$ be a path. Then $f_\gamma := X \times_S I \to I$ is a proper separated submersion, and by the previous result, the cohomology sheaves $R^q f_{\gamma_*} F$ are constant. By the proper base change theorem, $R^q f_{\gamma_*} F \cong \gamma^* \mathcal{F}^q$. Thus the result follows from Lemma 6.5. $\square$

References

[1] William Fulton. *Introduction to Toric Varieties.* Number 131 in Annals of Mathematics Studies. Princeton University Press, 1993.

[2] Marvin Greenberg. *Lectures on Algebraic Topology.* W. A. Benjamin, Inc., New York, 1967.

[3] Mark Gross and Bernd Siebert. Mirror symmetry via logarithmic degeneration data I. *Journal of Differential Geometry*, 72:169–338, February 2006.

[4] Alexander Grothendieck. *Groupes de Monodromie en Géométrie Algébrique (SGA7 I)*, volume 288 of *Lecture Notes in Mathematics.* Springer-Verlag, Berlin, Heidelberg, New York, 1972.

[5] Heisuke Hironaka. Triangulations of algebraic sets. In Robin Hartshorne, editor, *Algebraic Geometry Arcata 1974*, volume XXIX of *Proceedings of Symposia in Pure Mathematics*, pages 165–185, Providence, Rhode Island, 1975. American Mathematical Society.

[6] Luc Illusie, Kazuya Kato, and Chikara Nakayama. Quasi-unipotent logarithmic Riemann-Hilbert correspondences. *Journal of Mathematical Sciences, The University of Tokyo*, 12(1):1–66, 2005.

[7] Birger Iversen. *Cohomology of Sheaves.* Springer-Verlag, Berlin, Heidelberg, New York, 1986.

[8] Takeshi Kajiwara and Chikara Nakayama. Higher direct images of local systems in log Betti cohomology. *Journal of Mathematical Sciences, The University of Tokyo*, 15(2):291–323, 2008.

[9] Masaki Kashiwara and Pierre Schapira. *Sheaves on Manifolds*, volume 292 of *Grundlehren der Mathematischen Wissenschaften*. Springer-Verlag, Berlin, Heidelberg, New York, 1990.

[10] Kazuya Kato. Logarithmic structures of Fontaine-Illusie. In Jun-Ichi Igusa, editor, *Algebraic Analysis, Geometry, and Number Theory*, pages 191–224, Baltimore and London, 1989. Johns Hopkins University Press.

[11] Kazuya Kato and Chikara Nakayama. Log Betti cohomology, log étale cohomology, and log de Rham cohomology of log schemes over $\mathbf{C}$. *Kodai Math. J.*, 22(2):161–186, 1999.

[12] Arthur Ogus. Introduction to logarithmic algebraic geometry. in preparation, 180 pp.

[13] Arthur Ogus. Relatively coherent log structures. in preparation.

[14] Arthur Ogus. On the logarithmic Riemann-Hilbert correspondence. *Documenta Mathematica*, Extra Volume: Kazuya Kato's Fiftieth Birthday:655–724, 2003.

[15] Lior Pachter and Bernd Sturmfels, editors. *Algebraic Statistics for Computational Biology*. Cambridge University Press, The Pitt Building, Trumpington St., Cambridge, U.K., 2005.

[16] Laurence Siebenmann. Deformations of homeomorphisms on stratified sets. *Comm. Math. Helv.*, 47:123–163, 1972.

[17] Sampei Usui. Recovery of vanishing cycles by log geometry: Case of several variables. In *Proceedings of International Conference, Commutative Algebra and Algebraic Geometry and Computational Methods, Hanoi 1996*, pages 135–144. Springer Verlag, 1999.

[18] Sampei Usui. Recovery of vanishing cycles by log geometry. *Tohoku Math. J.*, 53(1):1–36, 2001.

[19] Jean-Louis Verdier. Dualité dans la cohomologie des espaces localement compacts. In *Séminaire Bourbaki*, volume 9. Soc. Math. Fr., 1964–1966.