Calculations in Deformation Theory

Nathan Owen Ilten

UC Berkeley

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Deformation Basics

Computing Versal Deformations

The Setup

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Basic idea: we deform X by perturbing the f_i.

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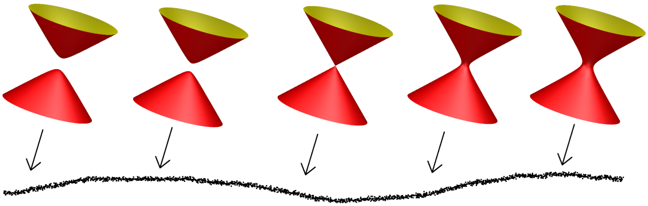
- The fiber over 0 is just X. The fiber over $t \neq 0$ is smooth.



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Problem: the relation $f_2 - f_1 = f_3$ doesn't lift to a relation among the $\widetilde{f_i}$.

Lifting Relations

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Definition

The relations R *lift* with respect to $\widetilde{F}$ subject to equations $g_1, \ldots, g_k \in T$ if there exists a $\widetilde{R} : \widetilde{S}^n \to \widetilde{S}^m$ restricting to R such that

$$\operatorname{Im}\left(\widetilde{F} \cdot \widetilde{R}\right) \subset \langle g_i \rangle$$

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$$\widetilde{F} = \left(\begin{array}{ccc} x_1x_3 - x_2^2 + tx_2 & x_2x_4 - x_3^2 - tx_4 & x_1x_4 - x_2x_3 \end{array} \right).$$

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- A lifting of R is given by $\widetilde{R}$.

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Example

For hypersurfaces, arbitrary perturbations are allowed.

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- Consider the deformation given by $\widetilde{f} = x^2 + y^2 - z^2 - t$.
- Can we induce the deformation given by $x^2 + y^2 - z^2 - sz$?
- Yes! Substitute $t = -\frac{1}{4}s^2$ and take the change of coordinates $z \mapsto (z + \frac{1}{2}s)$.

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A: Using `Macaulay2` and the package `VersalDeformations`.

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Basic approach: iteratively lift deformations in T^1_X to larger and larger base spaces.

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- $F = (\ x^2 + y^2 - z^2 \ )$.
- Any first order deformation can be induced from $F + t_1 \cdot 1 = (\ x^2 + y^2 - z^2 + t_1 \ )$ with $t_1^2 = 0$.

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These matrices solve the "deformation equation"

$$(\widetilde{F} \cdot \widetilde{R})^{\text{tr}} + C \cdot G = 0$$

where C is the sum of the list CL.

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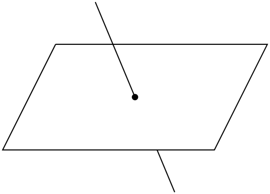
- The base space has two components, $\mathbb{C}^3$ and $\mathbb{C}$ meeting in a point.



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- Can calculate local (multigraded) Hilbert schemes.
- Can lift deformations in given tangent direction to a one-parameter family.

A Toric Fano Threefold

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Let X be the projective subscheme of $\mathbb{P}^8$ cut out by

$$\begin{aligned} x_{i+1}x_{i-1} - x_i y_0 & \qquad 1 \leq i \leq 6 \\ x_i x_{i+3} - y_0^2 & \qquad 1 \leq i \leq 3 \\ y_1 y_2 - y_0^2 & \end{aligned}$$

where $\mathbb{P}^8$ has coordinates $x_1, \ldots, x_6, y_0, y_1, y_2$.

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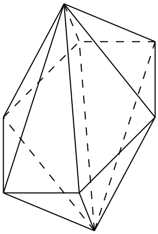

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Similar calculations + lots of hard work can be used to classify all smoothings of Gorenstein Fano toric threefolds of degree ≤ 12.

References

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T_X^1 may be computed as the cokernel of

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If $\dim \operatorname{Sing}(X) = 0$, then $\dim_{\mathbb{C}} T^1_X < \infty$.

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In general, this is not possible!!!

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3. C^0 is of the form $V \cdot D$, where $D \in \mathrm{Hom}(S^d, S^d)$ is a diagonal matrix.

The perturbation $\lim_{i\to\infty} F^i$ gives a formally versal deformation over the base space cut out by the rows of $\lim_{i\to\infty} G^{i-2}$.