C^*-algebras: Lecture Notes
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Office Hours: M 10-11:15, W 10:30-11:45, F 8:15-8:45
Recommended Text: Ken's book
Approximately 4 problem sets

1 January 18

2 January 20

Theorem 2.1. *Let A be a C^*-algebra. For any $a \in A^{sa}$, TFAE:*

1. *$a \in A^+$, i.e. $\sigma(a) \subset [0, \infty)$*
2. *$a = b^2$ for some $b \in A^{sa}$*
3. *$a = c^*c$ for some $c \in A$*

Furthermore, A^+ is a proper, closed, cone in A, i.e.

3 January 23

Proposition 3.1. *Let A be any associative algebra with 1, over $\mathbb{R}$ or $\mathbb{C}$, and let $a, b \in A$. Then $\sigma(ab) \cup \{0\} = \sigma(ba) \cup \{0\}$.*

4 January 25

From now on, an approximate identity for a C^*-algebra will be of norm one, consisting of positive elements.

5 January 27

We want to represent C^*-algebras on a Hilbert space - so we need positive linear functionals.

6 January 30

Let A be a C^*-algebra with 1. And let μ be a continuous positive linear functional on A.

$$\langle a, b\rangle_\mu = \mu(b^* a)$$

$$N_\mu = \{a : \mu(a^* a) = 0\}$$

with N_μ a left A-module. So $\angle\cdot,\cdot\rangle_\mu$ gives a definite inner product on A/N_μ. Complete $L^2(A,\mu)$. Left regular representations L of A on A drops to "*-representation" on A/N_μ.

Proposition 6.1. *For $a \in A$, $\|L_a\| \leq \|a\|$.*

Proof. Let $b \in A$,

$$\|L_a b\|_\mu^2 = \langle ab, ab\rangle_\mu = \mu(b^* a^* ab)$$

But $b^* a^* ab \leq \|a\|^2 b^* b$, so the above is

$$\leq \|a\|^2 \mu(b^* b) = \|a\|^2 \underbrace{\langle b, b\rangle_\mu}_{\|b\|_\mu^2}$$

□

Let $\xi_0 \in L^2(A,\mu)$ correspond to 1_A. Then ξ_0 is a cyclic vector, i.e. $\{L_a\xi_0 : a \in A\}$ is dense in $L^2(A,\mu)$. This is called the 'GNS' construction for Gelfand-Naimik-Segal.

Definition 6.2. For a normed *-algebra, a representation $(\pi, \underbrace{\mathcal{H}}_{Hilbert})$ is a cont *-homomorphism of A into $\mathcal{B}(CH)$. This is *non-degenerate* if $span\{\pi(a)\xi : a \in A,\ \xi \in \mathcal{H}\}$ is dense in $\mathcal{H}$. A vector ξ_0 is cyclic if $\{\pi(a)\xi_0 : a \in A\}$ is dense.

Definition 6.3. Given any $\xi \in \mathcal{H}$, $\|\xi\| = 1$, define

$$\mu_\xi(a) = \langle \pi(a)\xi, \xi\rangle_{\mathcal{H}},\ \text{positive},\ |\mu_\xi\| \leq 1$$

if representation is non-degenerate, $\|\mu_\xi\| = 1$. A state of form μ_ξ for $\xi \in \mathcal{H}$ for $(\pi,\mathcal{H})$ is called a *vector state* for $(\pi,\mathcal{H})$.

Given a state μ on a C^*-algebra A with 1, $L, L^2(A, mu)), \xi_0\tilde{1}_1$

$$\mu_{\xi_0}(a) = \langle L_a\xi_0, \xi_0\rangle_\mu = \mu(a)$$

So μ is a vector state for $(L, L^2(A,\mu))$.

Proposition 6.4. *Let $(\pi,\mathcal{H},\xi_0)$, $(\pi',\mathcal{H}',\xi_0')$ be cyclic representations. If $\mu_{\xi_0} = \mu_{\xi_0'}$, then there is a* unitary *$U : \mathcal{H} \to \mathcal{H}'$ with $U\xi_0 = \xi_0'$, U is an intertwining operator (A-module homomorphism) $U\pi(a) = \pi'(a)U$.*

Proof. For $a \in A$ set $U(\pi(a)\xi_0) = \pi'(a)\xi'$. It is well-defined and unitary. Use $\mu_{\xi_0} = \mu_{\xi_0'}$. □

Proposition 6.5. *Let A be a normed *-algebra with 1_A [with self-adjoint approximate identity of norm 1]. Then there is a natural bijection between:*

1. *The set of state of A*
2. *The unitary equivalence classes of pointed (have a specified cyclic vector) cyclic representation of A with cyclic vector of norm 1.*

Definition 6.6 (Direct sums of representations)**.** Let $(\pi_\lambda, \mathcal{H}_\lambda)_{\lambda \in \Lambda}$ be a family of *-representations of a *-algebra A, such that for each $a \in A \exists$ constant c_a with $\|\pi_\lambda(a)\| \leq c_a$ for all λ. Set

$$\mathcal{H} = \bigoplus_{\lambda \in \Lambda} \mathcal{H}_\lambda = \{\text{functions } \xi \text{ for } \Lambda \text{ to } \cup \mathcal{H}_\lambda \text{ such that } \xi_\lambda \in \mathcal{H}_{\lambda] \, \forall \lambda}\}$$

and

$$\sum_\Lambda \|\xi_\lambda\|^2 < \infty, \ \langle \xi, \eta \rangle_{\mathcal{H}} = \sum \langle \xi_\lambda, \eta_\lambda \rangle_{\mathcal{H}_\lambda}$$

Set $(\pi(a)\xi)_\lambda = \pi_\lambda(a)\xi_\lambda, \ \|\pi(a)\| \leq c_a$.

Definition 6.7. Let A be a C^*-algebra with 1_A. Then the *universal representation* of A is

$$\bigoplus_{\mu \in S(A)} (\pi_\mu, \underbrace{\mathcal{H}_\mu}_{\text{GNS}})$$

Properties: Every state of A is a vector state for this representation.

For every $a \in A$, $a = a^*$, and every $\lambda \in \sigma(a)$ there is a vector $\xi \in \mathcal{H}$ with $\|\xi\| = 1$.

$$\pi(a)\xi, \xi\rangle = \lambda$$

$\|\pi(a)\| \leq \|a\|$, but the above gives $\|\pi(a)\| = \|a\| \ \forall a \in A, \ a = a^*$. But for any $a \in A$,

$$\|\pi(a)\|^2 = \|\pi(a^*a)\| = \|a^*a\| = \|a\|^2$$

Suppose A is separable: Choose a countable subset E dense in A^{sa}. For each $a \in E$ choose $\lambda \in \sigma(a)$ $|\lambda| = \|a\|$ and choose state μ_a on A so that $\mu_a(a) = \lambda$. Then each $L^2(A, \mu_a)$ is separable and $\bigoplus_{a \in E} L^2(A, \mu_a)$ separable.

7 February 1

Let A be a normed *-algebra, without 1, and let μ be a continuous positive linear functional on A. Let $\tilde{A} = A$ with 1 adjoined. Extend μ to $\tilde{\mu}$ on $\tilde{A}$ by $\mu(1) = \|\mu\|$.

Want: $\tilde{\mu}$ is positive.

Lemma 7.1. *Let A have a bounded approximate identity of norm 1 (e_λ). Then:*

1. $\mu(a^*) = \overline{\mu(a)}$
2. $|\mu(a)|^2 \leq \|\mu\|\mu(a^*a)$

Proof. 1.

$$\mu(a^*) = \lim \mu(a^* e_\lambda) = \lim \langle e_\lambda, a\rangle_\mu$$
$$= \lim \overline{\langle a, e_\lambda\rangle_\mu} = \lim \overline{\mu(e_\lambda^* a)} = \overline{\mu(a)}$$

Counterexample: $\mathcal{P}_0$ = polys p with $p(0) = 0 \subset C_\infty((0,1])$. Define $\mu(p) = ip'(0)$. Not continuous.

2.

$$|\mu(ae_\lambda)|^2 = |\langle e_\lambda^*, a\rangle_\mu|^2 \leq \langle e_\lambda^*, e_\lambda\rangle_\mu \langle a, a\rangle_\mu$$
$$= \underbrace{\mu(e_\lambda e_\lambda^*)}_{\leq \|\mu\|} \mu(a^* a) \leq \|\mu\| \mu(a^* a)$$

□

Lemma 7.2. *Let A be a normed *-algebra, and let μ be a continuous positive linear functional. Define $\tilde{\mu}$ on $\tilde{A}$ by $\mu(1) = \|\mu\|$.* **If** *μ satisfies (1) and (2) of previous lemma, then $\tilde{\mu} \geq 0$.*

Proof.

$$\begin{aligned}
\tilde{\mu}\left((z1 + a)^*(z1 + a)\right) &= \tilde{\mu}(\bar{z}z1 + \bar{z}a + a^*z + a^*a) \\
&= |z|^2\|\mu\| + \underbrace{\mu(\bar{z}a) + \mu\left((\bar{z}a)^*\right)}_{2\Re(\bar{z}\mu(a))} + \mu(a^*a) \\
&\leq |z^2|\|\mu\| - 2|z||\phi(a)| + \mu(a^*a) \\
&\leq |z^2|\|\mu\| - 2|z|\sqrt{\|\mu\|\mu(a^*a)} + \mu(a^*a) \\
&= \left(|z|\|\mu\|^{\frac{1}{2}} - \sqrt{\mu(a^*a)}\right)^2 \geq 0
\end{aligned}$$

□

So the following theorem sums up what we wanted and have thus proved:

Theorem 7.3. *If A has approximate identity of norm 1, then $\tilde{\mu} \geq 0$.*

Theorem 7.4. *Let A be a normed *-algebra with approximate identity of norm 1. Let μ be a continuous positive linear functional, extend μ to $\tilde{\mu}$ on $\tilde{A}$ by $\tilde{\mu}(1) = \|\mu\|$. So $\tilde{\mu} \geq 0$. Form $L^2\left(\tilde{A}, \tilde{\mu}\right)$, let $\xi_0 = [1] \in L^2\left(\tilde{A}, \tilde{\mu}\right)$. Then when we restrict representation of $\tilde{A}$ to A, the representation is still non-degenerate (and $\mu(a) = \langle L_a\xi_0, \xi_0\rangle \; \forall a \in A$).*

Proof. To show that $\xi_0 \in$ closure of image of A in $L^2(A, \mu)$ $(= \{L_a\xi_0 : a \in A\})$. Choose $\{a_j\}$. $\|a_j\| \leq 1, \mu(a_j) \to \|\mu\|$. Claim that in $L^2\left(\tilde{A}, \tilde{\mu}\right)$, $a_j \mapsto \xi_0$.

$$\begin{aligned}
\langle \xi_0 - a, \xi_0 - a\rangle_{L^2\left(\tilde{A}, \tilde{\mu}\right)} &= \tilde{\mu}\left((1 - a)^*(1 - a)\right) \\
&= \underbrace{\overbrace{\tilde{\mu}(1)}^{\|\mu\|} - \mu(a_j)}_{\to 0} - \underbrace{\mu(a_j^*)}_{\to\|\mu\|} + \mu(a_j^* a_j) \to \leq 0
\end{aligned}$$

□

Let A be a normed *-algebra with 1. $S(A)$ is the state space $= \{\mu : \text{ positive } \mu(1) = 1 = \|\mu\|\}$. $S(A) \subset A'$.

Proposition 7.5. *$S(A)$ is a w^*-closed convex subset of A'.*

Proof. Convex: If $\mu\nu \in S(A)$, $t \in [0,1]$, then $t\mu + (1-t)\nu \in S(A)$. If $\{\mu_\lambda\}$ is a net in $S(A)$, $\mu_\lambda \overset{\text{weak}-*}{\longrightarrow} \nu \in A'$, then $\underbrace{\mu_\lambda(1)}_{=1} \to \nu(1)$ and $\underbrace{\mu_\lambda(a^*a)}_{\geq 0} \to \nu(a^*a) \geq 0$ with $\|\nu\| = 1$. □

From this we get $S(A)$ is weak-* compact.

For non-unital A,

$$S(A) = \{\mu \geq 0 : \|\mu\| = 1\}, \text{ Convex. Not } w^*\text{-closed}$$

$$\partial S(A) = \{\mu : \mu \geq 0, \|\mu\| \leq 1\} \text{ Convex, weak-* closed}$$

Any $\mu \in \partial S(A)$ is a convex combination of a $\nu \in S(A)$ and 0.

$$\mu = \|\mu\| \underbrace{\left(\frac{\mu}{\|\mu\|}\right)}_{\in S(A)} + (1 - \|\mu\|)0$$

For which $\mu \in S(A)$ is the GNS representation on $L^2(A, \mu)$ irreducible, i.e. no closed invariant subspaces?

8 February 3

Corollary 8.1. *For A normed *-algebra with approximate identity - $\{e_\lambda\}$, for μ a positive linear functional on A, $\lim \mu(e_\lambda) = \|\mu\|$.*

Proof. Well,

$$\lim \mu(e_\lambda) = \lim \angle e_\lambda \xi_{\tilde{\mu}}, \xi_{\tilde{\mu}}\rangle_{\tilde{\mu}} = \langle \xi_{\tilde{\mu}}, \xi_{\tilde{\mu}} rangle = \tilde{\mu}(1) = \|\mu\|.$$

□

Remember, $\underbrace{A\mathcal{H}_{\tilde{\mu}}}_{\text{linear span}}$ is dense in $\mathcal{H}_{\tilde{\mu}}$, so $e_\lambda a\xi \to a\xi$, and $e_\lambda \xi \to \xi$.

Definition 8.2. Let $(\pi, \mathcal{H})$ be a *-representation of a *-algebra A. A subspace $\underbrace{\mathcal{K}}_{\text{closed}} \subset \mathcal{H}$ is a π-invariant if $\pi(a)\xi \in \mathcal{K}$ for all $\xi \in \mathcal{K}$. (a submodule)

Note that if $\mathcal{H}$ is finite-dimensional, then $(\pi, \mathcal{H})$ 'is' the direct sum of a finite number of irreducible representations.

Proposition 8.3. *If $\mathcal{K}$ is invariant, then so is $\mathcal{K}^\perp$. So $\mathcal{H} = \mathcal{K} \oplus \mathcal{K}^\perp$, $\pi = (\pi \ or\ \mathcal{K}) \oplus (\pi \ or\ \mathcal{K}^\perp)$.*

Definition 8.4. $(\pi, \mathcal{H})$ is irreducible if $\mathcal{H}$ has no proper (closed) invariant subspaces. (simple module)

Examples include, $\mathcal{H} = L^2([0,1], \text{Lebesgue})$, $A = C([0,1])$.

Proposition 8.5. *Every non-degenerate *-representation is the direct sum of cyclic representations.*

Proof. 'Same' as for existence of orthonormal basis for a Hilbert space. □

If $\mathcal{K}$ is an invariant subspace of $\mathcal{H}$, $(\pi, \mathcal{H})$, and if P is the orthogonal projection of $\mathcal{H}$ onto $\mathcal{K}$, then $\pi(a)P = P\pi(a)$, i.e. P is an intertwining operator.

Notation: $End_A(\mathcal{H}) = \{T \in B(\mathcal{H}) : \pi(a)T = T\pi(a)\}$. i.e. $P \in End_A(\mathcal{H})$.

$End_A(\mathcal{H})$ is a von Neumann algebra, i.e. *-algebra of $B(\mathcal{H})$ closed for strong operator topology. If

$$\underbrace{\{T_\lambda\}}_{\subset End_A(\mathcal{H})} \xrightarrow{\text{s.o.}} T.$$

Then for $\xi \in \mathcal{H}$,

$$\pi(a)T\xi = \lim \pi(a)T_\lambda\xi = \lim T_\lambda\pi(a)\xi = T\pi(a)\xi.$$

so $T \in End_A(\mathcal{H})$.

Lemma 8.6 (Schur's lemma)**.** $(\pi, \mathcal{H})$ *is irreducible* $\iff$ $End_A(\mathcal{H}) = \mathbb{C}I$.

Proof. If $(\pi, \mathcal{H})$ not irreducible, then have $P \in End_A(\mathcal{H})$, so the equality fails. Conversely, suppose $End_A(\mathcal{H}) \neq \mathbb{C}I$, so $\exists T \in End_A(\mathcal{H})$, $T \notin \mathcal{C}I$. Take $\Re$ and $\Im$ part of T, so we can assume $T = T^*$. Consider $B = C^*(T, I)$ representative on $\mathcal{H}$.

For $C(\sigma(T))$, $\sigma(T)$ has more than one point.

In 1928, von Neumann had the double commutant theorem. Let N be a vN algebra, then $N = (N')'$, where $N = End_{N'}(\mathcal{H})$ and $N' = End_N\mathcal{H}$.

Definition 8.7. Let μ, ν be positive linear functionals. We write $\mu \geq \nu$ if $\mu - \nu \geq 0$ and say that ν is *dominated* by μ. Say that μ is *pure* if when $\mu \geq \nu$, $\nu = r\mu$, with $r \in [0,1]$.

Radon-Nikodyn theorem: Then there is an $h \in L^\infty(X,\mu)$, $1 \geq h \geq 0$, with $\nu(f) = \mu(fh)$ where h is the Radon-Nikodyn derivative. μ is pure $\iff$ $\mu = \delta_m$ for some $m \in M$.

Given $\mu \geq 0$, let $T \in End_A(\mathcal{H}_\mu)$, $0 \leq T \leq I$. Let

$$\begin{aligned}\nu_T(a) &= \langle \pi_\nu(a) T\xi_\mu, \xi_\mu\rangle_\mu \\ &= \langle T\mu(a)\xi_\mu, \xi_\mu\rangle \\ &= \langle \pi(a)T^{\frac{1}{2}}, T^{\frac{1}{2}}\xi_\mu\rangle \geq 0\end{aligned}$$

So $\mu(a^*a) - \nu(a^*a) = \langle \pi(a^*a)(I-T)\xi_\mu, \xi_\mu\rangle \geq 0$. So $\mu \geq \nu$.

□

9 February 6

Positive linear functionals μ, ν, with $\mu \geq \nu$. Given $T \in End_A(\mathcal{H}_\mu)$, $0 \leq T \leq I$, set $\nu_T(a) = \langle \pi_\mu(a)T\xi_{\tilde{\mu}}, \xi_{\tilde{\mu}}\rangle$, then $\mu \geq \nu_T \geq 0$.

Theorem 9.1. *The mapping $T \mapsto \nu_T$ is a bijection from $\{T \in End_A(\mathcal{H}_\mu) : 0 \leq T \leq I\}$ onto $\{\nu : \mu \geq \nu \geq 0\}$.*

Proof. Injective, if $\nu_T = \nu_S$, says $\nu_T(a) - \nu_S(a) = 0$. Then, if $a = bc$

$$0 = \langle \pi_\mu(a)(T-S)\xi_{\tilde{\mu}}, \xi_{\tilde{\mu}}\rangle = \langle (T-S)\pi(c)\xi_{\tilde{\mu}}, \pi(b^*)\xi_{\tilde{\mu}}\rangle \implies T - S = 0$$

Surjective: Given ν, $\mu \geq \nu \geq 0$, on A,

$$\begin{aligned}|\langle a,b\rangle_\nu|^2 &\leq \langle a,a\rangle_\nu \langle b,b\rangle_\nu \\ &= \nu(a^*a)\nu(b^*b) \leq \mu(a^*a)\mu(b^*b) \\ &= \langle a,a\rangle_\mu \langle b,b\rangle_\mu \\ &= \|\pi(a)\xi_{\tilde{\mu}}\|^2 \|\pi(b)\xi_{\tilde{\mu}}\|^2\end{aligned}$$

So $\langle a,b\rangle_\nu$ drops to $A/\mathcal{N}\mu$, ν is continuous and of 'norm' =1. So it extends to a non-negative continuous sesquilinnear form on $\mathcal{H}_\mu$, $[\cdot,\cdot]_\nu$. For any $\eta \in \mathcal{H}_\mu$, there is a vector, $T^*\eta$, in $\mathcal{H}_\mu$ with $[\zeta,\eta]_\nu = \langle \zeta, T^*\eta\rangle$ for all $\zeta \in \mathcal{H}$, $\eta \mapsto T^*\eta$ conjugate linear, norm ≤ 1, so T^* is a linear operator. $I \geq T \geq 0$,

$$\langle \pi_\mu(a)T\xi_{\tilde{\mu}}, \xi_{\tilde{\mu}}\rangle = \langle T\xi_{\tilde{\mu}}, \pi_\mu(a^*)\xi_{\tilde{\mu}}\rangle = \nu(a\ 1)$$

$$\langle T\pi_\mu(a)\xi_{\tilde{\mu}}, \pi(b)\xi_{\tilde{\mu}}\rangle = \nu(a)$$

which implies $T\pi(a) = \pi(a)T\forall a$. □

Definition 9.2. We say that μ is *pure* if $\mu \geq \nu \geq 0 \implies \nu = r\mu$, $r \in [0,1]$.

Theorem 9.3. *The GNS representation for μ is irreducible $\iff$ μ is pure.*

Proof. If GNS representation is not irreducible, there is proper invariant subspace, and the $\perp$ projection P is in $End_A(\mathcal{H}_\mu)$, and $\nu_P \notin r\mu$. If μ is not pure, so $\mu \geq \nu \geq 0$, with $\nu \neq r\mu$, then $T_\nu \in Hom_A(\mathcal{H}_\mu)$, $\neq rI$, so $End_A(\mathcal{H}_\mu) \neq \mathbb{C}I$, so by Schur's lemma, $(\pi_\mu, \mathcal{H}_\mu)$ is not irreducible. □

$S(A)$ is convex. (w^* compact if $1 \in A$).

Definition 9.4. Let C be a convex set in a vector space. A $v \in C$ is said to be an *extreme point* of C if $v = tw_1 + (1-t)w_2$ for $w_1, w_2 \in C$ and $t \neq 0, 1$, then $w_1 = v = w_2$.

Proposition 9.5. *For $\mu \in S(A)$, μ is extreme $\iff$ μ is pure.*

Proof. Suppose $\mu = t\nu_1 + (1-t)\nu_2$, $\nu_1, \nu_2 \in S(A)$. Then $\mu \geq t\nu_1$, $(1-t)\nu_2$. so if μ is pure, and if $t \neq 0$, $t\nu_1 = r\mu \implies \nu_1 = \mu$, and similarly, $\nu_2 = \mu$, so μ is extreme. Conversely, $\nu \neq 0, \neq \mu$, if $\mu \geq \nu$,

$$\mu = \|\nu\| \underbrace{\left(\frac{\nu}{\|\nu\|}\right)}_{\in S(A)} + \|\mu - \nu\| \underbrace{\left(\frac{\mu - \nu}{\|\mu - \nu\|}\right)}_{\in S(A)}$$

If $1 \in A$, $\|\nu\| = \nu(1)$, $\|\mu - \nu\| = \mu(1) - \nu(1)$, so $\|\nu\| + \|\mu - \nu\| = 1$, so the above equation is a convex combination for μ. If $1 \notin A$, $\|\nu\| = \lim \nu(e_\lambda)...$, so we get $= 1$. If μ is not pure, so $\nu \neq r\mu$, so $\frac{\nu}{\|\nu\|} \neq \mu$, then μ is not extreme. □

Theorem 9.6 (Krein-Milnor Theorem)**.** *Let C be a compact convex subset of a locally convex topological vector space. Then C is the closed convex hull of its set of extreme points.*

Proof. Sketch of proof: define a *face* of C to be a subset F of C such that if $v \in F$ and $v = tw_1 + (1-t)w_2$, $w_1, w_2 \in C$, $t \neq 0, 1$, then $w_1, w_2 \in F$. Given an continuous linear functional, ϕ, on V (over $\mathbb{R}$)

$$\{v \in C : \phi(v) = \max\{\phi(C)\}\}$$

is a closed face of C. Order the closed faces by decreasing inclusion. Use Hausdorff maximum principle to get minimal faces. They must be an extreme point. □

Definition 9.7. Let A be a *-algebra. Set

$$\|a\|_u = \|\pi_{\text{universal}}(a)\|$$

is a C^*-seminorm. The completion of A for this is a C^*-algebra, denoted $C^*(A)$, sometimes referred to a universal enveloping (C^*)-algebra for A. The continuous *-representations of A correspond to the *-representations of C^*(A).

Definition 9.8. Set

$$\|a\|_e = \sup\{\|\pi(a)\| : (\pi, \mathcal{H}) \text{ is an irreducible representation }\}.$$

The $\oplus$ of the $(\pi_\mu, \mathcal{H}_\mu)$, with μ extreme, is called the *atomic* representation of A.

10 February 8

Let A be a *-normed algebra. Let $a \in A, a = a^*$. Know:

$$\|a\|_u = sup\{|\mu(a)| : \mu \in S(A)\}. \quad (*)$$

Set $\|a\|_e = sup\{|\mu(a)| : \mu$ pure state of $A\}$. Need: $\|a\|_u = \|a\|_e$. Suppose that $\mu(a) \le c < \|a\|_u$ for all $\mu \in S_e(a)$, where e is a 'pure state'. Then for any μ which is a finite convex combination of elements of $S_e(A)$. Then $\mu(a) \le C$. Then if μ is the closure of $\mu(a) \le c$, i.e. $\mu(a) \le c$ for all $\mu \in S(A)$. Apply also to $-a$, i.e. if also $\mu(-a) \le c$ for $\mu \in S_e(A)$, which contradicts (*). Alternatively, we could have started with the fact $\|a\|_e = c < \|a\|_u$. So the representation of A on

$$\bigoplus_{\mu \in S_e(A)} (\pi_\mu, \mathcal{H}_\mu)$$

is faithful and gives norm $\| \cdot \|_u$. From last semester in the Banach algebra course, we have the following theorem:

Theorem 10.1 (Gelfand-Ractror 1943)**.** *Let G be a locally compact group. Then G* has lots of irreducible strongly continuous unitary representations*, i.e. given $f \in L^1(G)$, if $\pi(f) = 0$ for all irreducible unitary representations π, then $f = 0$.*

Proof. $L^1(G)$ has lots of states, namely all those coming from the left regular representation on $L^2(G)$, faithful on $L^1(G)$. □

If you were to take a very complicated finite group, it can be extremely difficult to list and exhibit all the irreducible representations. As soon as you have a specific group you have to use the specific details of that group to find its representation. For example, finding the specific representation of any Lie group, such as $\mathrm{SL}(n, \mathbb{R})$, can be very tedious.

Note that $\mathcal{B}_0(\mathcal{H})$ is (topologically) simple, i.e. no proper closed ideals. Sketch of proof: if you have a closed ideal then it has some non-zero operator in it. then you must have a non-zero vector taken to another. Then multiply your operator on both sides of those projections and your left with a rank one operator that has to be in the ideal. Then you can get all rank one operators in your ideal. But the set of the rank one operators are dense. The 'standard' representation of $\mathcal{B}_0(\mathcal{H})$ on $\mathcal{H}$ is irreducible. So there is no proper subspace that is invariant under the action of this algebra.

Theorem 10.2. *Let $A = \mathcal{B}_0(\mathcal{H})$ = algebra of compact operators on $\mathcal{H}$, a Hilbert space. Every (non-degenerate) representation of $\mathcal{B}_0(\mathcal{H})$ is unitarily equivalent to a direct sum of copies of the standard representation. In particular, up to unitary equivalence, $\mathcal{B}_0(\mathcal{H})$ has only one irreducible representation.*

Proof. Let $(\pi, \mathcal{V})$ be a *-representation of $\mathcal{B}_0(\mathcal{H})$, where $\mathcal{V}$ denotes a different Hilbert space, bad notation I know. Let inner products on $\mathcal{H}$ and $\mathcal{V}$ be linear in the second variable and put scalars on the right. For $\xi, \eta \in \mathcal{H}$, let $\langle \xi, \eta \rangle_c$ be the rank-one operator defined by

$$\langle \xi, \eta \rangle_c \zeta = \xi \langle \eta, \zeta \rangle_{\mathcal{H}}.$$

Now the inner product on the left is linear in the first variable. Also,

$$\langle \xi, \eta \rangle_c^* = \langle \eta, \xi \rangle_c \text{ and } T\langle \xi, \eta \rangle_c = \langle T\xi, \eta \rangle_c$$

where $T \in \mathcal{B}_0(\mathcal{H})$. Since $(\pi, \mathcal{V})$ is non-degenerate, you can find $\xi, \eta \in \mathcal{H}$ and $v \in \mathcal{V}$ such that $\pi(\langle \xi, \eta \rangle_c)v \neq 0$. For notation, we'll drop the π and just write $\langle \xi, \eta \rangle_c v$. We can certainly choose $\|\xi\| = 1$. Then

$$\langle \xi, \xi \rangle_c (\langle \xi, \eta \rangle_c v) = \langle \underbrace{\langle \xi, \xi \rangle_c \xi}_{=\xi}, \eta \rangle_c v = \langle \xi, \eta \rangle_c v$$

Replace v by $\langle \xi, \eta \rangle_c v$, so $\langle \xi, \xi \rangle_c v = v$ Assume that $\|v\| = 1$. Now define $\mathcal{Q} : \mathcal{H} \to \mathcal{V}$ by

$$\mathcal{Q}(\eta) = \langle \eta, \xi \rangle_c v.$$

The $\mathcal{Q}$ is isometric:

$$\begin{aligned} \langle \mathcal{Q}(\eta), \mathcal{Q}(\zeta) \rangle_{\mathcal{V}} &= \langle \langle \eta, \xi \rangle_c v, \langle \zeta, \xi \rangle_c v \rangle_{\mathcal{V}} \\ &= \langle \langle \xi, \xi \rangle_c \langle \eta, \xi \rangle_c v, v \rangle_{\mathcal{V}} \\ &= \langle \langle \langle \xi, \zeta \rangle_c \eta, \xi \rangle_c v, v \rangle_{\mathcal{V}} \\ &= \langle \langle \xi \langle \zeta, \eta \rangle_{\mathcal{H}}, \xi \rangle_c v, v \rangle_{\mathcal{V}} \\ &= \langle \eta, \zeta \rangle_{\mathcal{H}} \langle \underbrace{\langle \xi, \xi \rangle_c v}_{=v}, v \rangle = \langle \eta, \zeta \rangle_{\mathcal{H}} \end{aligned}$$

$\mathcal{Q}$ intertwines: Let $T \in \mathcal{B}_0(\mathcal{H})$. Then

$$\begin{aligned} \mathcal{Q}(T\eta) &= \langle T\eta, \xi \rangle_c v \\ &= (T \langle \eta, \xi \rangle_c)\, v \\ &= T\,(\langle \eta, \xi \rangle_c v) \\ &= T(\mathcal{Q}\eta) \end{aligned}$$

Now apply Zorn.

□

11 February 10

Notation: $\hat{A}$ is the 'set' of unitary equivalence class of irreducible representations. If you want to be very careful about the set theory, you take a Hilbert space of that cardinality and look at all the possible representations which will give you a set. We showed that $\mathcal{B}_0(\mathcal{H}))\hat{}$ has one element.

Theorem 11.1 (Naimark's conjecture)**.** *Let A be a C^*-algebra. If $\hat{A}$ has only one element, then $A \cong \mathcal{B}_0(\mathcal{H})$ for some $\mathcal{H}$.*

This is true if A is separable. In December 2003, Chuck Ahemann and Nick Weaver constructed a counterexample using the 'diamond principle' which is consistent with, but independent of, 'standard set theory'. So the above conjecture is irreducible for standard set theory. (these two were PhD students at Berkeley)

Definition 11.2 (Morita Equivalence)**.** Two unital rings R and S are *Morita equivalent* if their categories of left modules are equivariant.

Theorem 11.3 (Morita)**.** *This definition holds $\iff$ there is a bimodule ${}_RX_S$ with nice properties, then the equivalence of categories is: if ${}_SV \mapsto {}_RV_S \otimes_S {}_SV$.*

For A, B C^*-algebras, if you have ${}_AX_B$, then you need inner products $\langle\cdot,\cdot\rangle_A, \langle\cdot,\cdot\rangle_B$ for both ways where they generate all of A, resp. all of B. The nice thing is that this all works for non-unital C^*-algebras. A concrete example is when you have the bimodule $\mathcal{H}$ over $\mathcal{B}_0(\mathcal{H})$ and $\mathbb{C}$ with inner products $\langle\cdot,\cdot\rangle_c$ and respectively $\langle\cdot,\cdot\rangle_{\mathcal{H}}$. Then we can define $\langle x\otimes\xi, y\otimes\eta\rangle \stackrel{\text{def}}{=} \langle\langle y,x\rangle_B\xi,\eta\rangle_{\mathcal{H}}$, which satisfies the relation with respect to the first generic example of A and B,

$$\langle x,y\rangle_A z = x\langle y,z\rangle_B$$

A hope for Poisson manifolds is that the product of functions wants to deformed into a noncommutative product in the direction of the Poisson bracket. Even more, you'd like that this product give you a dense subalgebra of a C^*-algebra. Then maybe for Poisson manifolds there should be a Morita equivalence. Alan recently had a student that developed this concept of Morita equivalence on Poisson manifolds.

Every pure state (sometimes just called a 'state') μ of $\mathcal{B}_0(\mathcal{H})$ is a vector state for representations on $\mathcal{H}$, i.e. $\exists\xi\in\mathcal{H}: \|\xi\| = 1, \mu(T) = \langle T\xi,\xi\rangle$. $S_e(\mathcal{B}_0(\mathcal{H}))$ = set of rays, i.e. 1-dimensional subspaces of $\mathcal{H}$, $\sim P\mathcal{H}$ where P is a projection, or = set of rank-one projections on $\mathcal{H}$. Given rank-one projection P, $\mu_P(T) = \text{trace}(PT)$. For the entire phase space, in physics we call these density matrices $= S(\mathcal{B}_0\mathcal{H}) = \{D\geq 0, D \text{ of trace class}, \text{trace}(D) = 1\}$ with $\mu_D(T) = \text{trace}(DT)$.

Now a group of symmetries will act on the set of rays. This has strong implications for example if you have a group acting on a set of rays, that comes from either a unitary or anti-unitary operator. That means for each group element you can attach a unitary operator determined up to a scalar multiple of modulus one, and there's no convention to choose that scalar to just be one, thus the reason for projectors. If also you have a connected Lie group acting, then every element of the group has a square element in the group and since the square of any anti-unitary elements is a unitary element, as long as your group is connected then you just have unitary element. However, time reversal and parity and charge reversal are all given by anti-unitary operators in a Hilbert space. It's hard to assign to observables what operators you want to use, but this is generally what happens with projective operators in quantum physics. With semi-simple Lie groups, they always have a covering group that's simple connected and every projective unitary representation can be realized as a covering of the ordinary unitary representation. For example $SO(3)$ has covering $SU(2)$ with covering map being the adjoint representation. This is why $SU(2)$ is the group that arises with spin in quantum mechanics. For your group G, you can always find an extension group whose kernel is the circle group S^1 with the short exact sequence

$$0\to T\to E\to G\to 0$$

and E is your representation.

Moving on, we'd like to look at ideals and irreducible representations.

Proposition 11.4. *Let A be a C^*-algebra and let I be an ideal (closed, two-sided) in A. For any irreducible representation $(\pi,\mathcal{H})$ of A, either $pi(I) = \{0\}$ or $\pi\mid_I$ is irreducible.*

Proof. If $\pi\mid_I\neq\{0\}$, then look at $\overline{[\pi(I)\mathcal{H}]}$ which is a $\neq\{0\}$, A-invariant subspace of $\mathcal{H}$. Since our representation of A is supposed to be irreducible, then $\overline{[\pi(I)\mathcal{H}]} = \mathcal{H}$, i.e. representation of I on $\mathcal{H}$ is non-degenerate. But if $\mathcal{K}(\neq\{0\})\subset\mathcal{H}$ is I-invariant, then $\mathcal{K} = \overline{[\pi(I)\mathcal{K}]}$. So now suppose you have this property, then $\mathcal{K}$ is A-invariant, so $\mathcal{K} = \mathcal{H}$. □

Proposition 11.5. *Let I be an ideal in A. Let $(\pi,\mathcal{H}),(\rho,\mathcal{K})$ be two representations of A. If $\pi|_I,\rho|_I$ are non-degenerate, and if they are unitarily equivalent as representations of I, then they are unitarily equivalent as representations of A.*

Proof. Let $U : \mathcal{H} \to \mathcal{K}$ be an intertwining unitary map for $pi|_I$ and $\rho|_I$. Then for $a \in A, d \in I, \xi \in \mathcal{H}$,

$$\rho(a)U(\pi(d)\xi) = \rho(a)\rho(d)U\xi = \rho(\underbrace{ad}_{\in I})U\xi = U\pi(ad)\xi = U\pi(a)(\underbrace{\pi(d)\xi}_{\text{span } \mathcal{H}})$$

□

12 February 13

Theorem 12.1. *Let A be a C^*-algebra of $\mathcal{B}_0(\mathcal{H}), \dim \mathcal{H} \geq 2$ and suppose $\mathcal{H}$ is irreducible under the action of A. Then $A = \mathcal{B}_0(\mathcal{H})$.*

Proof. We can find $T \in A$ such that $T = T^*, T \neq 0$. $\sigma(T)$ is countable with only 0 as a limit point. By spectral theorem for compact self-adjoint operators, A will contain a projection $P \neq 0$, and P has finite rank. Choose P of minimal rank in A. For any $S \in A, S = S^*$, then $PSP \in A$ and is self-adjoint. The spectral projection of PSP are of rank $\neq \text{rank}(P)$. So the rank of these spectral projections $= \text{rank}(P)$. So $PSP = \alpha(S)P$ for some $\alpha(S) \in \mathbb{R}$. Then it's the same for any $S \in A$. Choose $\xi \in \mathcal{H}, P\xi = \xi$, $\|\xi\| = 1$. Let $\eta \in \mathcal{H}, P\eta = \eta$, with $\eta \perp \xi$. then for any $S \in A$,

$$\langle S\xi, \eta \rangle = \langle SP\xi, P\eta \rangle = \langle PSP\xi, \eta \rangle = \langle \alpha(S)\xi, \eta \rangle = 0$$

Since A acts irreducibly, $\overline{\{S\xi\}} = \mathcal{H}$. Thus $\eta = 0$. Thus P has rank 1. Now for any $S \in A$, $S\xi = SP\xi$, $R_n\xi = \eta_n \to \zeta$, for a dense set in $\mathcal{H}$, with SP of rank one. So A contains rank one operators carrying ξ to a dense set of vectors in $\mathcal{H}$. But A is closed. So A contains all rank one operators which are 0 or $\xi^\perp$. But A is closed under $*$. So A contains all rank one operators, so $A = \mathcal{B}_0(\mathcal{H})$. □

Theorem 12.2. *Let A be a C^*-algebra and let $(\pi, \mathcal{H})$ be an irreducible representation of A.*

1. *If $\pi(A)$ contains one non-zero compact operator, then $\pi(A) \supseteq \mathcal{V}_0(\mathcal{H})$.*
2. *Suppose $\pi(A) \supseteq \mathcal{B}_0(\mathcal{H})$. Let $I = \ker(\pi)$. then any other irreducible representation $(\rho, \mathcal{K})$ such that $\ker(\rho) = \ker(\pi)$ is unitarily equivalent to $(\pi, \mathcal{H})$.*

Proof. (1) Let $J = \pi^{-1}(\mathcal{B}_0(\mathcal{H}))$. J is a closed ideal in A, $\supset I$, $\neq I$ (if $\pi(A)$ contains one non-zero compact operator). Then $\pi\mid_J$ is an irreducible representation of J. So $\pi(J) \subseteq \mathcal{B}_0(\mathcal{H})$ and $\pi(J)$ acts irreducibly on $\mathcal{H}$. By previous theorem $\overline{\pi(J)} = \mathcal{B}_0(\mathcal{H}) = \pi(J)$. That proves (1).
(2) Let $(\rho, \mathcal{K})$ be irreducible and have $\ker(\rho) = \ker(pi)$. Then $\rho|_J$ representation of J, kernel I, so view ρ as a n irreducible representation of $J/I \overset{\pi}{\cong} \mathcal{B}_0(\mathcal{H})$. Similarly, $\pi|_J$ is irreducible representation of J/I. So there is a unitary operator $U : \mathcal{H} \to \mathcal{K}$ intertwining the representation π and ρ of J. But these representations of J ar non-degenerate. Thus U intertwines π and ρ as representations of A. □

Definition 12.3 (Kaplanski)**.** : Here CCR means 'completely contains representation'

1. A C^*-algebra A is CCR if for every $(\pi, \mathcal{H}) \in \hat{A}$. $\pi(A) = \mathcal{B}_0(\mathcal{H})$.
2. A C^*-algebra A is GCR if for every $(\pi, \mathcal{H}) \in \hat{A}$. $\pi(A) \supseteq \mathcal{B}_0(\mathcal{H})$.
3. A C^*-algebra A is NCR if for every $(\pi, \mathcal{H}) \in \hat{A}$. $\pi(A) \cap \mathcal{B}_0(\mathcal{H}) = \{0\}$.

CCR $\Rightarrow$ GCR.

If G is locally compact, we have $L^1(G)$. $C^*(G)$ is (full) group C^*-algebra of G obtained by completing the $L^1(G)$ with respect to a certain norm. $\|f\|_{C^*} = \sup\{\|\pi_f\| : (\pi, \mathcal{H})$ is a *-representation of $L^1(G)$. $\sim$ strongly continuous unitary representation of $G\}$, with $\|f\|_{C^*} \leq \|f\|_1$.

Theorem 12.4 (Haish-Chashar 1954). *Let G be a semi-simple Lie group. Then $C^*(G)$ is CCR.*

Theorem 12.5 (Dixmier 1957). *Let G be a nilpotent Lie group. Then $C^*(G)$ is CCR.*

The closed connected subgroups of uppertriangular matrices with ones on the diagonal, and their quotients by a discrete subgroup of the center.

13 February 15

Many solvable Lie groups have $C^*(G)$ being GCR but not CCR. Even more so, Mautner showed, (unpublished 1950's), that there are solvable Lie groups for which $C^*(G)$ is not GCR,
Mautner group: $\mathbb{C}^2$, define an action α of $\mathbb{R}$ on $\mathbb{C}^2$ by $\alpha_t(z_1, z_2) = (e^{2\pi it}z_1, e^{2\pi i\gamma t}z_2)$ where γ is irrational. Then $G = \mathbb{C}^2 \times_\alpha \mathbb{R}$. If you take any point on the torus and look at it's orbit under this action, then it winds around densely and never closes because of the irrational γ. This is like the picture we'll see when we get to non-commutative tori. Solvable Lie groups look like:

$$\sim G \subseteq \begin{pmatrix} \lambda_1 & & * \\ & \ddots & \\ 0 & & \lambda_n \end{pmatrix}, \ \lambda_j \in \mathbb{C} \setminus \{0\}.$$

Thorma (1964): A discrete group G has $C^*(G)$ GCR $\iff$ G has a normal Abelian subgroup A such that G/A is a finite group.

Definition 13.1. For a *-normed algebra, a *primitive* ideal is the kernel of an irreducible representation.

Theorem 13.2. *If a C^*-algebra A is GCR, then the map $\hat{A} \to \operatorname{Prim}(A)$, $(\pi, \mathcal{H}) \to \ker(\pi)$ is a bijection. If $\pi(A) \geq \mathcal{B}_0(\mathcal{H})$, then any two irreducible representations of A having kernel $= \ker(\pi)$ are unitarily equivalent.*

Theorem 13.3. *(many people, but Jim Glimm did hardest part in 1961) Let A be a C^*-algebra, and let $(\pi, \mathcal{H})$ be an irreducible representation of A, and let $I = \ker(\pi)$, (so $I \in \operatorname{Prim}(A)$). Suppose that $\pi(A) \cap \mathcal{B}_0(\mathcal{H}) = \{0\}$. Then there is an uncountable number of inequivalent irreducible representations of A all having I as kernel, and the set of all irreducible representations with I as kernel is unclassifiable.*

In the above situation, you can construct plenty of irreducible representations that have an ideal as the kernel, but you'll never be able to exhaust all the possibilities. This idea of unclassifiable is due to Mackey. A nice discussion of this is given in Pedersen's "C^*-algebras and their automorphism groups" (not a lot of examples and exercises).

For a locally compact G acting on locally compact X, look at orbits which are open in their closure (Good). Even further, if an orbit is closed, it's like the CCR, and if only open in their closure, then it's like the GCR situation. The Bad situation is when the orbits are not open in their closure.

Observation: Let A be a C^*-algebra with 1, infinite dimensional (purely as a linear space) and simple. Then A is not GCR. We're ruling out the matrix algebras.

For example, looking at 2x2 matrices,

$$M_2 \hookrightarrow M_4 \to M_8 \to M_{16} \to M_{32} \cdots$$

where

$$T \mapsto \begin{pmatrix} T & 0 \\ 0 & T \end{pmatrix} \cdots$$

Then you can look at the C^*-algebra $\overline{\cup M_{2^n}}$ which is simple. This is the CAR ("canonical anti-commutative relations"). Generally algebras, of the form $M_{n_1} \hookrightarrow M_{n_2} \to M_{n_3} \to \cdots$. The limits are the "UHF" (ultra hyper-finite) C^*-algebras (sometimes referred to as "Glimm algebras). Looking at the embedding $F_1 \hookrightarrow F_2 \hookrightarrow F_3 \cdots$ each F_i is a finite dimensional C^*-algebra called AF (approximately finite dimensional) C^*-algebra.

To wrap up, let's talk about what one would mean by unclassifiable with an example. The isomorphism classes of 7-dimensional real Lie algebras are unclassifiable. What does this mean? Take $\mathbb{R}^7$ and let $S =$ all bilinear maps $\mathbb{R}^7 \times \mathbb{R}^7 \to \mathbb{R}^7$ which give a nilpotent Lie algebra structure. Let $GL(7,\mathbb{R})$ act on S. The isomorphism classes of 7d nilpotent Lie algebras are $S/GL(7,\mathbb{R})$. Mackey said to look at the corresponding collection of Borel sets and give this the quotient Borel structure. Then we can prove that this is not countably separable. Given any two points in the quotients, we can find a Borel set that separates them. But you can't find a countable number of Borel sets that separate points.

14 February 22

The topology on $\text{Prim}(A)$ is T_0. Given $I \in \text{Prim}(A)$, $\overline{\{I\}} = \{J \in \text{Prim}(A) : I \subseteq J\}$. So $\{I\}$ is closed $\iff$ I is maximal. If $J \in \overline{\{I\}}$ and $I \in \overline{\{J\}}$ then $I = J$.

For a C^*-algebra:

1. $\text{Prim}(A)$ is locally compact (not necessarily Hausdorff). Every point has a neighborhood $\mathcal{N}$, every open cover of which has a finite subcover.

2. If A is also separable, $\text{Prim}(A)$ has a Baire category property.

Let $A = (\mathcal{B}_0(\mathcal{H}))^N$, $\text{Prim}(A) = \{\ \{0\} = I_0, \mathcal{B}_0(\mathcal{H}) = I_1\}$, $\overline{\{I_0\}} = \{I_0, I_1\}$, $\overline{\{I_1\}} = I_1$, i.e. $\{I_1\}$ is closed, $\{I_0\}$ is open.

Examples: Generators and relations.
Given a set of generators, $a_1, a_2, \ldots, a_n, \ldots$ (so put in generators $a_1^*, \ldots, a_n^*, \ldots$), and given a set of relations: Non-commutative polynomials in the a_j's and a_j^*'s. Equation $p = 0$.

Let $\mathcal{F}$ be a free algebra over $\mathbb{C}$ generated by the a_j's and the a_j^*'s. It is a *-algebra. Let $\mathcal{I}$ be the *-ideal generated by all the relations p. Set $A = \mathcal{F}/\mathcal{I}$. A is a *-algebra, generated by the a_j's and a_j^*'s, satisfying the relations $p = 0$.

Now you can look at (in principle) all $\pi : \mathcal{A} \to \mathcal{B}(\mathcal{H})$ *-homomorphisms. Set $\|a\| = \sup\{\|\pi(a)\| : (\pi, \mathcal{H})$ is 8-representation of $\mathcal{A}\}$. Notice that $\mathcal{A}$ completed for this norm is the universal C^*-algebra for the generators and relations.

1. Our first issue is that we need this to be finite. So we need that the relations force a finite upper bound for $\|\pi(a_j)\|_{\mathcal{B}(\mathcal{H})}$ for each generator. If it is not true, then there is no $C*$-algebra.

2. Show that there are *-representations other than the 0 representation. Otherwise, $C^*(\mathcal{A}) = \{0\}$, and we're certainly not interested.

3. Often: there is a natural "obvious" collection of *-representations which give a norm (or at least semi-norm), often a norm, which gives a C^*-algebra, but are not enough to give the universal C^*-algebra. e.g. G a discrete group, $C_C(G)$ left regular representation on $l^2(G)$, $C_r^*(G)$, universal $C^*(G)$, then $C^*(G) = C_r^*(G) \iff G$ is amenable. Another example, given generator $\{u\}$ and relation $uu^* = 1 = u^*u$, forces $\|u\|_{C^*}$. Then $C^*\{\ \{u\},\ \text{relations}\ \} \cong C(T)$, where T is the circle. Still another example, with generator $\{s\}$, and relation $s^*s = 1$. Then $\|s\|_{C^*} = 1$, then $C^*\{\ \{u\},\ \text{relations}\ \} \equiv C^*(\text{unilateral shift})$.

4. if you have the C^*-algebra from generators and relations, but how do you figure out it's structure?

Quantum $SU(2)$:

$$SU(2) = \left\{ x = \begin{pmatrix} \alpha & -\bar{\beta} \\ \beta & \bar{\alpha} \end{pmatrix} : \alpha, \beta \in \mathbb{C}, \alpha\bar{\alpha} + \beta\bar{\beta} = 1 \right\}$$

Topology is $S^3 \subset \mathbb{C}^2$. Define functions a, b on $SU(2)$ by $a(x) = \alpha$, $b(x) = \beta$. $a, b \in C(SU(2))$ commute, and $aa^* + bb^* = 1$. (The usual commutative C^*-algebra generated by a, b with $aa^* + bb^* = 1$, $\cong C(S^3)$) The group structure gives a comultiplication: $\Delta : A \to A \otimes A \sim C(G \times G)$. Then $f \in C(G)$, $(\Delta f)(x, y) \overset{\text{def}}{=} f(xy)$.

$$m : A \otimes A \to A, \ \ (\Delta a)(x, y) = a(xy) = \alpha\alpha' - \bar{\beta}\beta' = a(x)a(y) - b^*(x)b(y)$$

with

$$y = \begin{pmatrix} \alpha' & -\bar{\beta}' \\ \beta' & \bar{\alpha}' \end{pmatrix}$$

So we see that $\Delta a = a \otimes a - b^* \otimes b$.

Woronovicz in 1986 was at the University of Warsaw, just after he discovered all this, during the solidarity movement and downfall of the Soviet Union, he had the misfortune of being nominated dean of his faculty so it delayed his work.

Even to this day, people don't know how to define a quantum group other than with generators and relations.

15 February 24

Quantum SU(2):

- $SU(2)$: Generators a, b, a^*, b^* commute, have a *-algebra for these generators called $A \subset C(SU(2))$ with $a^*a + b^*b = 1$.
- $\Delta : A \to A \otimes A$ comultiplication, coassociative Δa, with $(\Delta f)(x, y) \stackrel{\text{def}}{=} f(xy)$.
- coidentity element $\varepsilon : A \to \mathbb{C}$
- coinverses $S : A \longrightarrow A$

In Quantum groups, we want A to be non-commutative, but we still have Δ, ε, S. This is closely related to Hopf algebras. Choose $q \in (0, 1]$ with q" $='' e^{\hbar}$. For $SU_q(2)$, have generators a, b, a^*, b^* and relations $ab = qba$, $bb^* = b^*b$, and $ab^* = qb^*a$ but we still have $a^*a + b^*b = 1$, with $aa^* + q^2bb^* = 1$. Now if we're taking the representations of the commutative algebra, $C(SU(2))$, that's very different from taking the representations of the group. Worono—- showed that the representation of the quantum group version is very closely related to classical representation.

We'd like to have a more geometrical representation for this, like $C^\infty(SU(2))$, and have some sort of twisting, but we still can't get past generators and relations. Now there is a very extensive theory of compact quantum groups with Haar measure, with much left to be done. For locally compact groups, we have the difficulty that if you take a non-compact group like matrices, and look at the entries, you get unbounded functions that correspond to unbounded operators. No one has proved that Haar measure exists on the locally compact groups. Once you take Haar measure existence as one of the axioms, the general theory works out. (recall that compact for a quantum group means whether or not it has an identity element). Furthermore, suppose I have one generator, a, that's self-adjoint, i.e. $a = a^*$, then we don't really have a C^*-algebra. However, from the point of view of spectral theory, we can look at the C^*-algebra $C^*(a, a = a^* = C_\infty(\mathbb{R})$ under certain conditions, which actually just looks like $a(t) = t$. We say a is "affiliated" with $C_\infty(\mathbb{R})$. Now you can certain compatify, but recognize that compacting something like the real line, you no longer have a group.

If you try to work this out for a finite group G, and you have $A, m, i, \Delta, \varepsilon$, the dual A' swaps everything so that m becomes Δ, i becomes ε, etc. If you work this out where $A = C(G)$, then the dual is simply $A' = l^1(G)$ and multiplication becomes convolution, comultiplication becomes $\Delta(\delta_x) = \delta_x \otimes \delta_x$. The universal enveloping algebra kind of looks more like the dual. Increasing in math and physics, groups arose from symmetries, and quantum groups gives us certain symmetries. In the same way a group can act on a space or functions on a space, a quantum group can act on an algebra.

Now let G be a discrete group.
generators: all elements of G
relations: all relations between elements of G, i.e. $xy = z$, and $x^* = x^{-1}$ for all $x \in G$.
Then the C^*-algebra of the above is simply $C * (G)$.

The next class of examples are tensor products of C^*-algebras. Let A and B be unital C^*-algebras. We have

$$a \leftarrow a \otimes 1_B, \quad b \to 1_A \otimes b$$

and

$$(a \otimes 1_B)(1_A \otimes b) = a \otimes b = (1_A \otimes b)(a \otimes 1_B)$$

Generators: $A \cup B$
Relations: all relations between elements of A and all relations between elements of B, with $ab = ba\ \forall a \in A, b \in B$ and $1_A = 1_B$. Also, $\alpha a \otimes b = a \otimes \alpha b$.

Define * in the evident way, i.e. $(a \otimes b)^* = a^* \otimes b^*$. We get therefore that $\mathcal{F}/\mathcal{I} = A \otimes B$ is a *-algebra over $\mathbb{C}$.

Are there any *-representations?
"Obvious representations": Let $(\pi, \mathcal{H})$ be a representation of A, and $(\rho, \mathcal{K})$ a representation of B.
For example, on the algebraic tensor product $\mathcal{H} \otimes_{\text{alg}} \mathcal{K}$, we have the inner product

$$\langle \xi \otimes \eta, \xi' \otimes \eta' \rangle \stackrel{\text{def}}{=} \langle \xi, \xi' \rangle \langle \eta, \eta' \rangle$$

Extend by sesqui-linearity. Complete $\mathcal{H} \otimes_{\text{alg}} \mathcal{K}$ for the inner product.
For the normal tensor product $\mathcal{H} \otimes \mathcal{K}$, we have

$$\pi \otimes \rho : A \otimes B \longrightarrow \mathcal{B}(\mathcal{H} \otimes \mathcal{K}), \text{ with } S \otimes T = (S \otimes I_{\mathcal{K}})(I_{\mathcal{H}} \otimes T)$$

and $(\pi \otimes \rho)(a \otimes b) = \pi(a) \otimes \rho(b)$ and extend by linearity.

For any *-representation σ of $A \otimes B$, set $\sigma^A(a) = \sigma(a \otimes 1_B)$. So we get a *-representation of A, and $\|\sigma^A(a)\| \leq \|a\|$. This certainly means that a C^*-algebra exists. Now for $t \in A \otimes_{\text{alg}} B$, set

$$\|t\|_{\min} = \sup\{\|(\pi \otimes \rho)(t)\| : \pi \text{ is *-representation of } A \text{ and } \rho \text{ is a *-representation of } B\}.$$

16 February 27

Given $t \in A \otimes_{\text{alg}} B$, then

$$\begin{aligned} \|t\|_{\min} &= \sup\{\| \underbrace{(\pi \otimes \rho)}_{\mathcal{H}\otimes\mathcal{K}}(t)\| : (\pi, \mathcal{H}) \text{ is a representation of } A,\ (\rho, \mathcal{K}) \text{ is a representation of } B\} \\ \|t\|_{\max} &= \sup\{\|\pi \otimes \rho\| : \ \pi \text{ and } \rho \text{ are commuting representations of } A \text{ and } B \text{ on a Hilbert space} \\ &\qquad \mathcal{H},\ \pi(a)\rho(b) = \rho(b)\pi(a)\ \ \forall a, b\} \end{aligned}$$

E.g.: For any locally compact group G, have left regular representation λ of $C_r^*(b)$ on $L^2(G)$, integrated form of $(\lambda_x \xi)(y) = \xi(x^{-1}y)$, and have right regular representation of $C_r^*(b)$ on $L^2(G)$, integrated form of $(\rho_x \xi)(y) = \xi(yx)$. Gives a representation $\lambda \otimes \rho$ of $C_r^*(G) \otimes_{\max} C_r^*(G)$.

1959 Takesaki: For $G = F_2$, the representation $\lambda \otimes \rho$ is not continuous for $\|\cdot\|_{\min}$.

Definition 16.1. A C^*-algebra A is *nuclear* if $A \otimes_{\min} B = A \otimes_{\max} B$ for all C^*-algebras B.

Theorem 16.2. *For G discrete $C_r^*(G)$ is nuclear $\iff$ G is amenable. (Fails for G not discrete, e.g. $SL(2, \mathbb{R})$)*

If B does not have 1,

$$A \otimes_{\text{alg}} B \subset \tilde{A} \otimes \tilde{B}$$

$A \subset A \otimes B$, with $a \mapsto a \times 1_{\mathcal{M}(A)}$ and $A \hookrightarrow \mathcal{M}(A \otimes B)$.

Can do *free products* of C^*-algebras, $A * B$, in same way. (note: require a and b commute) For example, $C(S^1) * C(S^1) \cong C^*(F_2)$, which come from $C^*(u : uu^* = 1 = u^*u)$ and $C^*(v : vv^* = 1 = v^*v)$.

C^*-dynamical systems (discrete):

Have G discrete group, A a C^*-algebra, and $\alpha : G \to \text{Aut}(A)$. E.g. A is Abelian. i.e. $A = C_\infty(M)$, with M locally compact. Given function $\alpha : G \to \text{Homeo}(M)$ where M represents the dynamical systems. Then define $(\alpha_x(f))(m) \stackrel{\text{def}}{=} f({\alpha_x}^{-1}(m))$. Originally this came from quantum systems where G are your symmetries and the algebra is the quantum algebra. The earliest papers were written by physicists. They wanted all of this construction working on one Hilbert space, thus the following definition:

Definition 16.3. A *covariant representation* of (A, G, α) is a $(\pi, U, \mathcal{H})$ where π is a representation of A on $\mathcal{H}$ and U is a representation of G on $\mathcal{H}$. We require that $\pi(\alpha_x(a)) = U_x \pi(a) {U_x}^{-1}$, called "the covariance relation".

You can look for the C^*-algebra generated by: $A \cup G$, with relations: All the ones of A, all the ones of G, * of A, $x^* = x^{-1}$, $\alpha_x(a) = xax^{-1}$. Yes, it always exists, though it may be zero.

Take $C^*(A, G, \alpha)$, $A \times_\alpha G$, called the covariance algebra or crossed-product algebra. Is $A \times_\alpha G \neq \{0\}$?

One more comment: the words in elements of A and x, with $xa = \alpha_x(a)x$, can always be reduced to the form ax. So the algebraic algebra has as elements sums of the form

$$\sum_{x \in G} f(x)x, \ f \in C_C(G, A)$$

where compact support here means finite support. It's a matter of convention which side you put the x, but most people seem to put the x's on the right.

"Obvious" covariant representations: Let $(\rho, \mathcal{H}_0)$ be any representation of A. Let $\mathcal{H} = l^2(G, \mathcal{H}_0)$. Let U be defined by $(U_x\xi)(y) = \xi(x^{-1}y)$. $\xi \in \mathcal{H}$. Then

$$(\pi(a)\xi)(y) = \rho(\alpha_y^{-1}(a))(\xi(y))$$

The reason for doing it this way is to really see if it satisfies the covariance condition.

$$((U_x(\pi(a))\xi)(y) = (\pi(a)\xi)(x^{-1}y) = \rho(\alpha^{-1}_{x^{-1}y}(a))\xi(x^{-1}y) = \rho(\alpha_{y^{-1}}(\alpha_x(a)))(\xi(x^{-1}y)) = (\pi(a)(U_x\xi))(y)$$

The covariant representation *induced* from $(\rho, \mathcal{H}_0)$ of A. Given $H \subset G$, and a covariant representation $(\rho, V, \mathcal{H}_0)$ of $(A, H, \alpha|_H)$,, we can *induce* this to a covariant representation of (G, A, α). The case above is $H = \{e_G\}$. (This is in $\mathcal{H} = l^2(G/H, \mathcal{H}_0)$. For $\sum f(x)x$, any covariant representation $(\pi, U, \mathcal{H})$, set $\sigma_f = \sum \pi(f(x))U_x$ an operation on $\mathcal{H}$. This will give a representation of $A \times_\alpha G$

17 March 6

G-invariant ideals in $C_\infty(M)$ correspond to G-invariant closed (or open) subsets of M.

Theorem 17.1. *Assume that M has a countable basis for its topology. Let $(\sigma, \mathcal{H})$ be an irreducible representation of $C_\infty(M) \times_\alpha G$ which is the integrated form of $(\pi, U, \mathcal{H})$. Let $I = \ker \pi$, so I is a G-invariant ideal in $C_\infty(M)$. Let $Z_I = \text{hull}(I) = \{m : \varphi(m) = 0\ \forall \varphi \in I\}$. Then Z_I is the closure of a G-orbit in M, i.e $\exists m_0$ such that $Z_I = \overline{\{\alpha_x(m_0) : x \in G\}}$.*

Proof. Choose a countable basis for the topology of M. Let $\{B_n\}$ be an enumeration of these basis elements which meet Z_I. (So the $(B_n \cap Z_I)$'s form a basis for the topology of Z_I). Let $O_n = \alpha_G(B_n)$, an open G-invariant subset of M, meeting Z_I. Let $J_n = C_\infty(O_n) = \{\varphi \in C_\infty(M) : \varphi|_{O'_n} \equiv 0\} \subseteq C_\infty(M)$. Then J_n is a G-invariant ideal in $C_\infty(M)$. $J_n \times_\alpha G$ is ideal in $C_\infty(M) \times_\alpha G$, and since $O_n \cap Z_I \neq \emptyset$, $\sigma|_{J_n \times_\alpha G} \neq 0$ representation. Since σ is irreducible, $\sigma|_{J_n \times_\alpha G}$ is non-degenerate, so $pi|_{J_n}$ is a non-degenerate representation of J_n on $\mathcal{H}$. Choose $\xi \in \mathcal{H}$, $\|\xi\| = 1$. Define μ on M by $\mu(\varphi) = \langle \pi(\varphi)\xi, \xi\rangle_{\mathcal{H}}$, which is a vector state. Let μ also denote the corresponding probability Borel measure on M. (in particular we care about the measure of sets $\mu(O_n)$). For a given n, choose in J_n a non-negative approximate identity $\{e^n_k\}$ of norm 1, then $\langle \pi(e_k)\xi, \xi\rangle \to \langle \xi, \xi\rangle = 1$, and on the other hand, this is the same as $\mu(e_k)$, where $e_k \in C_\infty(M)$, have norm ≤ 1, and are supported on O_n. Then $\mu(O_n) = 1$. Thus $\mu(O'_n) = 0$. So $\mu(\cup^\infty_n O'_n) = 0$. Thus $\mu(\cap^\infty O_n) = 1$. Then $\cap^\infty O_n \neq \emptyset$. Let $m_0 \in \cap O_n$ (but actually, $m_0 \in Z_I$ as well, since $\cap O_n \subseteq Z_I$. Given $m_1 \notin Z_I$, $\exists B_n \in$ base for topology of M, $m_1 \in B_n$, $B_n \cap Z_I \neq \emptyset$ and $O_n = \alpha_G(B_n)$...will finish explaining next time). For any $m \in Z_I$, any neighborhood $\mathcal{N}$ of m, $\exists n$ so $m \in B_n \cap Z_I \subset \mathcal{N}$. Then $m_0 \in O_n = \alpha_G(B_n)$, so $\exists x_0 \in G$ so $\alpha_{x_0}(m_0) \in B_n$, so $\alpha_G(m_0) \in B_n \cap Z_I \subset \mathcal{N}$. So $\alpha_G(m_0)$ is dense in Z_I. □

E.g. let $M = T$, the circle in $\mathbb{C}$. Pick θ to be an irrational number. Let α be an action of $\mathbb{Z} \in G$ by α_n a rotation by angle $2\pi\theta$. then every orbit is dense and every orbit is countable.

18 March 8

Last time we had: I, Z_I, $\{B_n\}$, $O_n = \alpha_G(B_n)$, with $\mu(O_n) = 1$. If $f \in C_C(M)$ is 0 on Z_I, then $f \in I$, so $\mu(f) = 0$, thus $\mu(Z_I) = 1$. Thus $\mu(O_n \cap Z_I) = 1$ for all n. Thus $\mu(\bigcap(O_n \cap Z_I)) = 1$. So $\exists m_0$ in the intersection so $m_0 \in Z_I$. for each n, $m_0 \in O_n = \alpha_G(B_n)$ so $\exists x_n \in G$, $\alpha_{x_n}(m_0) \in B_n \cap Z_I$. This intersection forms a base for the topology of Z_I. Thus $\alpha_G(m_0)$ is dense.

Theorem 18.1. *(Pretty fundamental) Let G be a locally compact group. Let $M = G$ with α action by left translation. Then $C_\infty(M) \times_\alpha G = \mathcal{B}_0(L^2(G))$.*

Proof. We first have the evident covariant representation $(\pi, Y, L^2(G))$ (Schrodinger representation), with $(\pi(\varphi)\xi)(x) = \varphi(x)\xi(x)$ and $(U_y\xi)(x) = \xi(y^{-1}x)$ for $\varphi \in C_\infty(M), \xi \in L^2(G)$. We can check the covariant condition. Let σ be its integrated form. For $f \in C_C(G, C_\infty(G)) \sim C(G \times M)$.

$$(\sigma_f\xi)(x) = (\pi(f(y))U_y\xi)(x) = \int f(f, x)\xi(y^{-1}x)dy$$

For $\varphi, \psi \in C_C(G) \subset L^2(G)$, let

$$\langle \varphi, \psi\rangle_C \xi = \varphi\langle \psi, \xi\rangle_{L^2(G)}$$

i.e.

$$\begin{aligned}
\left(\langle\varphi,\psi\rangle_C\xi\right)(x) &= \varphi(x)\int\bar{\psi}(y)\xi(y)dy \\
&= \varphi(x)\int\bar{\psi}(y^{-1})\xi(y^{-1})\Delta(y^{-1})dy \\
&= \int\varphi(x)\bar{\psi}(y^{-1}x)\Delta(y^{-1}x)\xi(y^{-1}x)dy
\end{aligned}$$

where

$$\langle\varphi,\psi\rangle_E(y,x)=\varphi(x)\bar{\psi}(y^{-1}x)\Delta(y^{-1}x)$$

The point is that we want to get rank one operators in $C_\infty(M)\times_\alpha G$. Suppose you can find an f that doesn't take the value zero ($\sigma(f)=0$ then $f\equiv 0$), then you can cook up a ξ such that this integral is non-zero. So σ is a faithful representation of that uncompleted *-algebra. So, we get that $\sigma_{\langle\varphi,\psi\rangle_E}=\langle\varphi,\psi\rangle_C$, the algebra of the LHS corresponds exactly to the algebra of the RHS. We can also see that given any two vectors in $C_C(G)$, which is dense in L^2, we can approximate them as close as we want by elements which are continuous with compact support, so the claim is that this representation, σ, is irreducible. (In the operator norm closure you get all rank one operators). Let E be the linear span of the $\langle\varphi,\psi\rangle_E$'s for $\varphi,\psi\in C_C(G)$. So

$$\langle\varphi,\psi\rangle_E\langle\varphi',\psi'\rangle_E=\langle\varphi,\psi'\rangle_E\langle\varphi',\psi\rangle_{\mathcal{L}^2(G)}$$

Thus E is a *-subalgebra of $C_C(G,C_{\infty(M)})$ with the * operation.
Claim: E is a dense +-subalgebra, for $\|\cdot\|_1$, thus for $\|\cdot\|_{C^*}$.

For E is closed under point-wise multiplication.

$$\begin{aligned}
\langle\varphi,\psi\rangle_E(y,x)\langle\varphi',\psi'\rangle_E(y,x) &= \varphi(x)\bar{\psi}(y^{-1}x)\Delta(y^{-1}x)\varphi'(x)\bar{\psi}'(y]invx)\Delta(y^{-1}x) \\
&= \underbrace{(\varphi\varphi')(x)}_{\in C_C(G)}\underbrace{(\psi\psi'\Delta)(y^{-1}x)}_{\in C_C(G)}\Delta(y^{-1}x)
\end{aligned}$$

and E is closed under complex conjugation, and separates points, thus E is dense in $C_C(G\times G)$ for the inductive limit topology. Thus dense in $C_C(G,C_\infty(G))$ for

$$\|f\|_1=\int_G\|f(y)\|_\infty dy=\int\sup_{x\in G}|f(y,x)|dy$$

Thus E is dense in $C_\infty(G)\times G$. Thus at this point, it's clear that $\sigma(C_\infty(G)\times_\alpha G)\subset\mathcal{B}_0(L^2(G))$. Why does this have anything to do with the C^* norm? The sneaky thing to do is the following observation. Let $\{varphi_1,...,\varphi_n\}$ be in $C_C(G)$, orthonormal. Then the linear span of $\{\langle\varphi_j,\varphi_k\rangle_C\}$, which will act exactly like matrix units, will be $\cong M_n(\mathbb{C})$. Thus in $C_C(G,C_\infty(G))$, the linear span of $\{\langle\varphi_j,\varphi_k\rangle_E\}\cong M_n(\mathbb{C})$, which is a complete C^*-algebra, so its C^*-norm is unique. So on $\{\langle\varphi_j,\varphi_k\rangle_E\}$, the norm from $C^*(G,C_\infty(G),\alpha)$ must agree with the norm from σ. But any element of E is contained in one of these $M_n(\mathbb{C})$'s. So on E, the $C^*(C_\infty(G),G,\alpha)$-norm agrees with norm from σ. So completions are the same, i.e. completion of E for $\sigma\cong\mathcal{B}_0(L^2(G))$.

□

19 March 17

Have $G \xrightarrow{\pi} \mathrm{Aut}(\mathcal{PH})$. We can find $U : G \to \mathcal{U}(\mathcal{H})$, so that $\pi_x = (U_x$ acting on $\mathcal{PH})$. But U_x only unique up to a scalar,

$$U_x U_y = c(x,y)U_{xy} \text{ such that } c : G \times G \to T$$

For some important examples it is impossible to choose c so it is continuous. If G, H are separable, we can choose c to be measurable.
From associativity: $c(xy,z)c(x,y) = c(x,yz)c(y,z)$ a 2-cocycle on G with values in T. We can choose $U_e = I_{\mathcal{H}}$. Then $c(x,e) = 1 = c(e,x)$, "normalized cocycle".

On $L^1(G)$, define $*_c$ to be the *twisted convolution*

$$(f *_c g)(x) = \int_G f(y)g(y^{-1}x)c(y, y^{-1}x)dy$$

with $\|f *_c g\|_1 \leq \|f\|_1\|g\|_1$, and $f^{*_c}(x) = \overline{f(x^{-1})}c(x,x^{-1})$.

Now you can look at $C^*(G,c)$ as the completion of $L^1(G,c)$ for a C^*-norm from all $*$-representations. Define representations on $L^2(G)$ by $f *_c \xi =$ same formula, with $\xi \in L^2(G)$. (Another way of viewing this is that with ordinary convolutions, you can ask what forms you can put in to get an associative product.) This gives a faithful representation, $C^*_r(G,c)$.

If c and c' are cohomologous, the $C^*(G,c) \cong C^*(G,c')$. Given an extention

$$0 \to T(\text{or } A) \to E \to G \to 0$$

with a map $\sigma : G \to T$, then the cocycle $c_\sigma(x,y) = \sigma(xy)\sigma(x)^{-1}\sigma(y)^{-1} = \partial\sigma(x,y)$ measures the difference in σ. If you can the Haar measure on G and Haar measure on T to get a measure on $T \times G$, and look only at the Borel structure and Haar measure, you get what is referred to as a measurable group that has an invariant measure. The a theorem of Andre Weyl from 1940 says that any group that is not genuinely locally a compact group can't have an invariant measure. Given G and Weyl representation W with the cocyle c_W and you look at the C^*-algebra that you get from this, $C^*(G,c_W)$, then this is naturally isomorphic to the compact operators $\mathcal{K}(L^2(G))$. It's interesting because this group starts off as an abelian group and you get this phenomenon with compact operators, like $G := \mathbb{R}^d$.

Now consider projective representations of $\mathbb{Z}^d$. Choose cocycles of the form: choose $\theta \in M_d(R)$, and set $c_\theta(m,n) = e^{2\pi i m \cot(\theta_n)} = e(m \cdot \theta_n)$, where $e(t) := e^{2\pi i t}$, is a bicharacter on $G = \mathbb{Z}^d$. So we can consider $C^*(\mathbb{Z}^d, c_\theta)$. If $\theta \equiv 0$, we get $C^*(\mathbb{Z}^d) \cong C(T^d)$. We then call this the non-commutative, or quantum, tori. For $G = \mathbb{Z}^d$, $\hat{G} = T^d$, can define an action α, "the dual action" of $\hat{G}$ on $C^*(G,c)$ by

$$(\alpha_x(f))(n) := \underbrace{\langle n, x\rangle}_{\in T} f(n)$$

It's clear that $\|\alpha_x(f)\|_1 = \|f\|_1$ and you can check that $\alpha_x \in \mathrm{Aut}(L^1(G,c))$. For $\hat{G} \to \mathrm{Aut}(L^1(G))$, $x \mapsto \alpha_x$ is a group homomorphism. This action is strongly continuous. For G discrete, just comes from

$$\alpha_x(\delta_n) = \langle n, x\rangle \delta_n$$

so it's continuous on $C_C(G)$, dense and α_x is isometric.

For G abelian, α an action of G on A, have a C^*-algebra $A \times_\alpha G$. Have dual action $\hat{\alpha}$, of $\hat{G}$, defined by $(\alpha_x(f))(n) = \langle n, x\rangle f(n)$, with $f \in C_C(G, A)$.

Let α be any action of T^d (or any compact abelian group (and compact group G)) on a C^*-algebra A, e.g. $A = C^*(\mathbb{Z}^d, c_\theta)$. Can do Fourier analysis. For $n \in \hat{G}(= \mathbb{Z}^d)$, and for $a \in A$ set $a_n = \int_G \overline{\langle x, n\rangle}\alpha_x(a)dx \in A$.

$$\begin{aligned}\alpha_y(a_n) &= \alpha_y\left(\int \overline{\langle x, n\rangle}\alpha_x(a)dx\right)\\ &= \int \overline{\langle x, n\rangle}\alpha_{y+x}(a)dx\\ &= \int \overline{\langle x+y, n\rangle}\alpha_x(n)dx\\ &= \langle y, n\rangle a_n\end{aligned}$$

where $\langle -y, n\rangle = \langle y, n\rangle^{-1} = \overline{\langle y, n\rangle}$

Set $A_n = \{a \in A : \alpha_x(a) = \langle x, n\rangle a\}$. Thus $a_n \in A_n$, "the n-isotypic component". Want to think of $A = C(T^d)$ where A_n is the one-dimensional subspace spanned by the function $x \mapsto \langle x, n\rangle$.

20 March 20

Have G compact Abelian, α an action (strongly continuous) of G on a Banach space (Frechet space). For $n \in \hat{G}$, and $a \in A$ set $a_n = \int_G \overline{\langle x, n\rangle}\alpha_x(a)dx$. Then $\alpha_y(a_n) = \langle y, n\rangle a_n$, let $A_n = \{a \in A : \alpha_y(a) = \langle y, n\rangle a \forall y \in G\}$ closed and linear. Set $e_n \in C(G) \subseteq L^1(G)$, by $e_n(x) = \overline{\langle x, n\rangle}$. Then the above equality for a_n gives $a_n = \alpha_{e_n}(a)$, such that $\alpha_{e_n}\alpha_{e_m} = \alpha_{e_n * e_m}$. Then

$$(e_m * e_n)(x) = \int e_m(y)e_n(x-y)dy = \int \overline{\langle y, m\rangle\langle x-y, n\rangle}dy = \overline{\langle x, y\rangle}\int_G \overline{\langle y, m-n\rangle}dy$$

$$\left(\text{since } y\int_G 1 = 1 \text{ we have}\right) = \overline{\langle x, n\rangle}\delta_{mn} = e_m\delta_{mn}$$

If $a \in A_m \cap A_n$, then $a = \alpha_{e_m}(a) = \alpha_{e_m}(\alpha_{e_n}(a)) = 0$. So $A_m \cap A_n = \{0\}$ if $m \neq n$.

The A_m's are linearly independent. You can form $\bigoplus_{n\in\hat{G}} A_n \subset A$. It's natural to ask how big the subspace is, and we have the following proposition.

Proposition 20.1. *The above subspace is dense in A.*

Proof. The linear span of the e_n's in $C(G)$ is closed under point-wise multiplication, complex conjugation and separates points. (we did this enough last semester knowing you have a Haar measure on a compact abelian group.) So by Stone-Weierstrass, it's dense for $\|\cdot\|_\infty$, so it is for the $\|\cdot\|_1$. We can form an approximate identity g_k consisting of finite sums of e_n's, then $\alpha_{g_k}(a) \to a$ and each one is in $\bigoplus A_m$ which shows the density. (note that there's really no way to say anything about the orthogonality of the spaces). □

Now if A is a Banach *-algebra, and if α is a *-algebra automorphism (i.e. each α_x is). Given $a, b \in A$, then you can look at what α_x does to a_nb_m

$$\alpha_x(a_nb_m) = \alpha_x(a_n)\alpha_x(b_m) = \overline{\langle x, n\rangle\langle x, m\rangle}a_nb_m = \overline{\langle x, n+m\rangle}a_nb_m$$

This shows a grading structure such that $A_nA_m \subseteq A_{m+n}$. Then

$$\alpha_x(a_n^*) = (\alpha_x(a_n))^* = (\overline{\langle x, n\rangle} a_n)^* = \overline{\langle x, -n\rangle} a_n^*$$

So $(A_n)^* = A_{-n}$.

We can do this sort of thing for $\hat{G}$ discrete, not abelian, that you call "C^*-algebraic bundles." In the non-discrete case, you'd have to show that the fibers (A_n) form a field. Then you can take continuous cross-sections of support and convolve them. The technical difficulty of saying all this is the word "bundle". We won't get into this description. However, something worth pointing out that if you look at A_0, it is a *-subalgebra.

So we are going to be dealing with this structure and want to get into the topic of smooth structure. But there's another generalization of getting into locally compact groupoids. Certainly, we can look at specifically T^n with the Lie group structure. However, we'll look at any general Lie group G. So we can consider closed connected subgroups, G, of $GL_n(\mathbb{R})$. Setting $\mathfrak{g} = \{X \in M_n(\mathbb{R}) : \forall t \in \mathbb{R}, \exp(tX) \in G\}$, we have some major theorems like $\mathfrak{g}$ is a Lie subalgebra of $M_n(\mathbb{R})$, i.e. $X, Y \in \mathfrak{g} \implies [X, Y] := XY - YX \in \mathfrak{g}$. Also that in a neighborhood $0 \in \mathfrak{g}$, exp is a diffeomorphism onto a neighborhood of $e_G = I$. Very important for our purposes, for each $X \in \mathfrak{g}$, $t \mapsto \exp(tX)$ is a 1-parameter subgroup of G.

So let G be any Lie group. Let α be a strongly continuous action (if your Lie group is not compact, you don't need that the action be isometric or bounded, i.e. $\|\alpha_x\| \leq K\ \forall x$) of G on a Banach space A (or Frechet space), then for $X \in \mathfrak{g}$, $a \in A$, you can ask is a differentiable in the X-direction, i.e. is $x \mapsto \alpha_x(a)$ differentiable at 0 in the X-direction? In ordinary calculus, you can spell it out to ask does

$$\lim_{t\to 0} \frac{\alpha_{\exp(tX)}(a) - a}{t} \text{ exist (in } A\text{, for } \|\cdot\|)?$$

If this exist for all X's, one would say it's once differentiable. Then if you have this much, denote the limit by $\mathcal{D}_X a$. Then you can ask, does $\mathcal{D}_Y(\mathcal{D}_X a)$ exists for all $X, Y \in \mathfrak{g}$? If so, then you can say that your element is twice differentiable, and so on. Then set

$$A^\infty = \{a : \mathcal{D}_{X_1} \ldots \mathcal{D}_{X_k} a \text{ exists } \forall k \geq 1,\ \forall X_1, ..., X_k \in \mathfrak{g}\}$$

The point is that A^∞ is a linear subspace of A (not necessarily closed).

Theorem 20.2. *A^∞ is dense in A.*

Proof. (Gärding) For any $f \in C_C^\infty(G)$ and for any $a \in A$, with $\alpha_f(a) \in A^\infty$. This sort of argument is sometimes call smoothing. We'll continue this proof next time. □

21 March 22

So rephrasing the theorem:

Theorem 21.1. *Let α be an action of a Lie group G on a Banach (or Frechet) space A (not necessarily uniquely bounded). Then A^∞ is dense in A.*

Proof. Claim: for any $f \in C_C^\infty(G)$, $\alpha_f(a) \in A^\infty$. Once have that, let f_n run through an approximate identity for $L^1(G)$, i.e. an approximate δ-function at e_G. Then $\alpha_{f_n}(a) \to a$, so A^∞ is dense.

Proof of claim: Let $X \in \mathfrak{g}$. Then

$$\begin{aligned}
\frac{\alpha_{\exp(tX)}(\alpha_f(a)) - \alpha_f(a)}{t} &= \frac{1}{t}\left(\alpha_{\exp(tX)}\int f(y)\alpha_y(a)dy - \int f(y)\alpha_y(a)dy\right) \\
&= \frac{1}{t}\left(\int f(y)\alpha_{(\exp(tX))y}(a)dy - \int f(y)\alpha_y(a)dy\right) \\
&= \frac{1}{t}\left(\int f(\exp(-tX)y)\alpha_y(a)dy - \int f(y)\alpha_y(a)dy\right) \\
&= \int\left(\frac{f(\exp(-tX)y) - f(y)}{t}\right)\alpha_y(a)dy \\
\left(\text{inserting } \lim_{t\to 0}\right) \quad &= \int(\mathcal{D}_x f)(y)\alpha_y(a)dy
\end{aligned}$$

Note: you can Taylor expand in t,

$$f(\exp(-tX)y) = f(y) + (\mathcal{D}_X f)(y)t + \frac{(\mathcal{D}_X^2 f)(y_t)}{2}t^2 + \cdots$$

so then we get

$$\frac{f(\exp(-tX)y) - f(y)}{t} - (\mathcal{D}_X f)(y) = \frac{\mathcal{D}_X^2 f(y_t)}{2}t$$

and we get a bound $|\cdot|$ on the LHS to be $\leq \frac{1}{2}\|\mathcal{D}_X^2 f(y)\|t$.

To finish the proof, we have that $\mathcal{D}_X(\alpha_f(a))$ exists and is $\alpha_{\mathcal{D}_X f}(a)$, then $\mathcal{D}_Y(\mathcal{D}_X\alpha_f(a)) = \alpha_{\mathcal{D}_Y\mathcal{D}_X f}(a)$ so in fact, continuing this process we can see that it's infinitely differentiable, so we're done. □

Definition 21.2. Linear span of $\{\alpha_f(a) : f \in C_C^\infty(G), a \in A\}$ is called the *Gärding domain.*

But is it equal to A^∞? (we can ask all these questions for G acting on $C_\infty(G)$) Then we have the following theorem:

Theorem 21.3 (Diximer-Malhaven)**.** *For $C_C^\infty(G)$ with convolution and for certain G's, connected Lie groups, there are $f \in C_C^\infty(G)$ which can not be expressed as $g * h$ for $g, h \in C_C^\infty(G)$. But finite linear combinations always work.*

Let α be an action of T^d on a Banach space A. For $n \in \mathbb{Z}^d$, $a \in A$, we defined $a_n = \alpha_{e_n}(a)$, $e_n(x) = \overline{\langle x, n\rangle}$. We had $A_n = \{a : \alpha_x(a) = \overline{\langle x, n\rangle}a\}$ and that $\bigoplus A_n$ is dense in A. Thus,

Corollary 21.4 (Riemann-Lebesgue Lemma)**.** *For any $a \in A$, $n \mapsto \|a_n\|$ is in $C_\infty(\mathbb{Z}^d)$.*

Proof. For $a \in \bigoplus A_n$, $n \mapsto \|a_n\|$ has finite support. For any $a, b \in A$, $\|a_n - b_n\| = \|(a-b)_n\| \leq \|a - b\|$. □

Now let's move on to this differentiability stuff. Then letting $G = T^d$, given $X \in \mathfrak{g} = \mathbb{R}^d$, $\exp(X) = \mathrm{im}(X) \in T^d = \mathbb{R}^d/\mathbb{Z}^d$. If $a \in A_n$, then

$$\mathcal{D}_X(a) = \lim\frac{\alpha_{\exp(-tX)}(a) - a}{t} = \lim\frac{\langle tX, n\rangle a - a}{t} = \lim\frac{e^{2\pi i tX\cdot n} - 1}{t}a = (2\pi i X\cdot n)a$$

with $\mathcal{D}_y\mathcal{D}_X a = (2\pi i Y \cdot n)(2\pi i X \cdot n)a$. Thus $A_n \subset A^\infty$, so $\bigoplus A_n \subset A^\infty$, and the infinitely differentiable elements are dense.

Often, when you get to this situation we write α_X for $\mathcal{D}_X$ We'll switch now to this notation.

Given $a \in A^\infty$, $X \in \mathfrak{g}$, what is $\alpha_X(a))_n$? Well this is $\alpha_{e_n}(\alpha_X(a))$. For any $f \in C^\infty(T^n)$, what is $\alpha_f(\alpha_X(a))$? This is just

$$\begin{aligned}\lim_{t\to 0}\alpha_f\left(\frac{\alpha_{tX}(a)-a}{t}\right) &= \lim \frac{1}{t}\left(\int f(y)\underbrace{\alpha_y(\alpha_{tX}(a))}_{\alpha_{y+tX}(a)}dy - \int f(y)\alpha_y(a)dy\right)\\ &= \lim\frac{1}{t}(-------) = \lim\frac{1}{t}\left(\int\frac{f(y-tX)-f(y)}{t}\alpha_y(a)dy\right)\\ &= \alpha_{-\mathcal{D}_X f}(a)\end{aligned}$$

and we get the relation $\alpha_f(\mathcal{D}_X a) = \alpha_{-D_X f}(a)$.

Similar things apply for compact Lie groups and instead of characters we use other representations.

Just to wrap it up, for $a \in A^\infty$, we have

$$(\mathcal{D}_X a)_n = \alpha_{e_n}(\mathcal{D}_X a) = \alpha_{-\mathcal{D}_X e_n}(a) = -2\pi i X \cdot n a_n$$

We'll see next time that the coefficient a_n die faster.

22 March 24

So we have the T^d action α on a Banach space A. For $a \in A$, we have $a_n \in A_n$ and $\mathcal{D}_X a$ for $X \in \mathfrak{g}$.
$(\mathcal{D}_X a)_n = 2\pi i\langle n, X\rangle a_n$

$$\|2\pi i\langle n, X\rangle a_n\| = \|\mathcal{D}_X a\|_n \le \|\mathcal{D}_X a\|$$

Use the standard inner product on $\mathfrak{g} = \mathbb{R}^d$ (or some other inner product allows you to choose a Riemannian metric, or a norm on $\mathfrak{g}$ leads to Finsler geometry). Choose orthonormal basis, e.g. standard basis, $E_j \in \mathfrak{g}$.

Aside: For $\mathfrak{g} = \mathbb{R}^d$, $T^d = G$, then $\mathcal{D}_X\mathcal{D}_y = \mathcal{D}_Y\mathcal{D}_X$ (or $\mathcal{D}_X\mathcal{D}_Y - \mathcal{D}_Y\mathcal{D}_X = \mathcal{D}_{[X,Y]}$).
Now $(\mathcal{D}^2_{E_j}a)_n =)2\pi i)^2\langle n, X\rangle^2 a_n$. Set the Laplacian of a to be $\Delta a = -\sum \mathcal{D}^2_{E_j}a$. For $a \in A^\infty$,

$$(\Delta a)_n = -(2\pi i)^2\underbrace{\sum\langle n, E_j\rangle^2}_{\sum n_j^2 = n\cdot n} a_n$$

and we end up with

$$((I+\Delta)a)_n = (1+4\pi^2 n\cdot n)a_n$$

and

$$\|(I+\Delta)a\| \ge \|((I-\Delta)a)_n\| \ge (1+4\pi^2 n\cdot n)\|a_n\|,$$

so

$$\|a_n\| \le (1+4\pi^2 n\cdot n)^{-1}\|(I+\Delta)\|.$$

Now we can iterate to get

$$((I+\Delta)^p a)_n = (1+4\pi^2 n\cdot n)^p a_n,$$

then

$$\|a_n\| \le (1+4\pi^2 n\cdot n)^{-p}\|(I+\Delta)^p a\|, \ \ \forall p\ge 1 \text{ for } a\in A^\infty$$

Thus, for any polynomial P in $n\in\mathbb{Z}^d$ variables, and you take $p > \deg(P)$, that means that $|P(n)\|a_n\|\ |$ is bounded in n. By definition of $\mathcal{S}(\mathbb{Z}^d,\{A_n\})$, $\{a_n\}$ is in here. This is simply the counterpart of the well-known fact that if you look at $C^\infty(T^d) \overset{\text{Fourier}}{\longleftrightarrow} \mathcal{S}(\mathbb{Z}^d)$, and our goal would be to show this. There is no characterization of which functions in $C_\infty(\mathbb{Z}^d)$ arise as Fourier transforms in $C(T^d)$ for $C(T^d)\hookrightarrow C_\infty(\mathbb{Z}^d)$. But what we're about to see for the infinitely differentiable elements in A, there is a very nice characterization, which is the Schwartz space.

For sufficiently large p, depending on d, the map $n\mapsto (1+4\pi^2 n\cdot n)^{-p}\in l^1(\mathbb{Z}^d)$. (I think that all you need is $p/2 > d$).

Then for p satisfying the above, we have that $P(n)(1+4\pi^2 n\cdot n)^{p+1}\in l^1(\mathbb{Z}^d)$, for all linear polynomials P, i.e. sums of the form $\langle n, X\rangle$. So for $a\in A^\infty$, any sequence of the form $\{\|\langle n,X\rangle a_n\|\}\in l^1(\mathbb{Z}^d)$. So suppose that we have any sequence $\{a_n\}\in\mathcal{S}(\mathbb{Z}^d,\{A_n\})$. Then $\{\|a_n\|\}\in l^1(\mathbb{Z}^d)$, so $a=\sum a_n$ is well-defined. For any given $X\in\mathfrak{g}$, $\sum\|2\pi i\langle n,X\rangle a_n\|\in l^1(\mathbb{Z}^d)$, so $b_X=\sum 2\pi i\langle n,X\rangle a_n$ is well-defined. Then we claim that $\mathcal{D}_X a$ exists and is b_X. Well, to show this, all we need to do is look at

$$\lim_{t\to 0}\left(\frac{\alpha_{\exp tX}(a)-a}{t}-b_X\right)\overset{?}{=}0$$

Well then we have this becomes

$$\lim_{t\to 0}\left(\frac{\sum\alpha_{\exp tX}(a_n)-a_n}{t}-\sum 2\pi i\langle n,X\rangle a_n\right) = \lim_{t\to 0}\sum\left(\underbrace{\frac{e(n\cdot tX)-1}{t}}_{\text{bounded}}-2\pi i\langle n,X\rangle\right)a_n$$

Given $\epsilon>0$, can choose N so that $\|\sum_{\|n\|\ge N}^{\infty}(\text{ above })\|<\epsilon/2$. Then

$$\|\sum_n(\text{ “above” })\| \le \|\sum_{\|n\|\le N}(\ \ ''\ \ )\| + \|\sum_{\|n\|>N}\|.$$

Then we can find a $\delta>0$ such that if $|t|<\delta$, then $\|\sum_{\|n\|\le N}(\ \ ''\ \ )\|<\epsilon/2$. Because of this rapid decrease, all of this can be iterated. So we get the following theorem:

Theorem 22.1. *The map $a\mapsto\{a_n\}$ gives a bijection between A^∞ and $\mathcal{S}(\mathbb{Z}^d,\{A_n\})$.*

There's been a lot of interest recently in the fact that if you take $SO(n)$ acting on S^{n-1}, with $n\ge 4$, you have a copy of T^2 in $SO(n)$. What we'll see is that for $C_C(\mathbb{Z}^d,c_\theta)$ we've perturbed the ordinary product by a non-commutative product and people have found in recent years that they have very interesting properties that mimic the properties of spheres.

23 April 3

24 April 5

25 April 7

$A_\theta = C(T_\theta)$, action α of $\mathbb{T}^d$ on A_θ, the dual action.

Let G be a compact group, α action of G on some A

$$A^\alpha := \{a : \alpha_x(a) = a \forall x \in G\}$$

Define $P : A \to A$, $Pa = \int_G \alpha_x(a)dx$ " $= a_0''$, $\int_G 1 = 1$.

Properties:

1. $Pa \in A^\alpha$. For $a \in A^\alpha$, $Pa = a$. So $P^2 = P$.
2. $a > 0$, then $Pa > 0$. If $b, c \in A^\alpha$, $a \in A$, then $P(bac) = bP(a)c$
3. $P(\alpha_g(a)) = P(a)$ "P is α-invariant."

P is a "conditional expectation" of A onto A^α. If $P : A \to B \subseteq A$ is a conditional expectation, then

$$\langle a_1, a_2 \rangle_B = P(a_1^* a_2)$$

Suppose I is an α-invariant ideal in A. Then for $d \in I$, $d > 0$, then $P(d) \in I$, $P(d) \neq 0$. So $I \cap A^\alpha \supset \{0\}$.

For A_θ, $a \in A_\theta$, $\alpha_x(a_n) = \langle n, x \rangle a_n$ where $a_n = \int \overline{\langle n, x \rangle} \alpha_x(a)dx$.

Definition 25.1. An action α of a compact group G on a unital C^*-algebra A, is *ergodic* if $A^\alpha = \mathbb{C}1_A$.

Proposition 25.2. *The action of $\mathbb{T}^d$ on A_θ is ergodic.*

Proposition 25.3. *Let α be ergodic action of a compact group on a unital C^*-algebra A. Then A has no proper α-invariant ideals (even non-closed).*

Proof. If you have a non-closed proper invariant ideal, it's closure is invariant as well. So we only need to look at closed ideals. For $I \cap A^\alpha \supset \{0\}$, so $1_A \in I$. □

Another ergodic action would be some G action on $C(G/H)$. There are definitely some unsolved questions in this area.

Theorem 25.4 (Olsen-Pedersen-Takesaki)**.** *Every ergodic action of $\mathbb{T}^d$ is on an A_θ.*

An open question would be: What are the ergodic actions of $SU(3)$ (or $SU(n)$, $n \geq 3$)? For $SU(2)$ it's known, due to Anthony Wassermann, but nothing exciting happens.

Proposition 25.5. *For ergodic action ..., $P(a) = \varphi(a)1_A$. φ is a state, is α-invariant (because P is), and in the only α-invariant state on A.*

Proof. If μ is α-invariant state on A, then

$$\mu(a) = \int_G \mu(\alpha_x(a))dx = \mu(\int_G \alpha_x(a)dx) = \mu(P(a)) = \mu(\varphi(a)1_A) = \varphi(a).$$

□

Going back to Mackey, you can construct many irreducible representations, but you just can't classify them.

Now we'd like to describe the center of A_θ. Notice, $a \in \text{Center}(A_\theta) \iff \delta_n a = a\delta_n$ for all $n \in \mathbb{Z}^d$. What δ_m's $\in \text{Center}(A_\theta)$?

$$\begin{aligned}
\delta_n\delta_m = \delta_{n+m}c_\theta(n,m) &= \overline{c_\theta(m,n)}c_\theta(n,m) \\
&= \overline{e(m\cdot\theta_n)}e(n\cdot\theta_m) \\
&= e(n\cdot(\theta-\theta^t)m) \\
&= \overline{e(\theta^t m\cdot n)} \\
&= e(-n\cdot\theta_m)
\end{aligned}$$

The key relation is $\delta_n\delta_m = \rho_\theta(n,m)\delta_m\delta_n$. We can rewrite this as a conjugation: $\delta_n\delta_m{\delta_n}^{-1} = \rho_\theta(n,m)\delta_m$. Thus $\delta_m \in \text{Center}(A_\theta) \iff \rho_\theta(n,m) = 1\forall n \in \mathbb{Z}^d$. Let

$$D_\theta = \{m \in \mathbb{Z}^d : \delta_m \in \text{Center}(A_\theta)\} = \{m \in \mathbb{Z}^d : \rho_\theta(n,m) = 1\forall n \in \mathbb{Z}^d\}$$

For each $n \in \mathbb{Z}^d$, $m \mapsto \rho_\theta(n,m)$ is in $\hat{\mathbb{Z}}^d = \mathbb{T}^d$, $\sim \delta_n = (\theta-\theta^t)n \in \mathbb{R}^d$ viewed in $\mathbb{R}^d/\mathbb{Z}^d = \mathbb{T}^d$.

So $\gamma : \mathbb{Z}^d \to \mathbb{T}^d$ is a group homomorphism determined by θ. So it gives you some subgroup, not necessarily closed. Let $H =$ closure of $\{\delta_m \in n \in \mathbb{Z}^d\}$ is $\mathbb{T}^d$. The point is that H acts on the subgroup A_θ via $Qa = \int_H \alpha_S(a)ds$.

We'll continue this in a week, and for the next two lectures, we'll here about K-theory.

26 April 14

A_θ a non-commutative torus. $\theta \in M_d(\mathbb{R})$. α of T^d on A_θ, $\delta_m *_\theta \delta_n * \delta_{-m} = \rho_\theta(m,n)\delta_n = \alpha_{\gamma_m}(\delta_n)$, with

$$\rho_{theta}(m,n) = c_{\theta-\theta^t}(m,n) = e^{2\pi i m\cdot(\theta-\theta^t)n}$$

For fixed m, this is a homomorphism of $\mathbb{Z}^d \to T$, call it γ_m, $m,n \in \mathbb{Z}^d$. Then $\gamma : \mathbb{Z}^d \to T^d$, where $\gamma_m = (\theta-\theta^t)m \in \mathbb{R}^d/\mathbb{Z}^d$. So we have the relation,

$$\gamma_m \in Z(A_\theta) \iff n \in D_\theta = \{n : \rho_\theta(m,n) = 1\forall m \in \mathbb{Z}^d\} = \{n : \alpha_{\gamma_m}(\delta_n) = \delta_n \forall m \in \mathbb{Z}^d\}$$

Let $H = \overline{\{\gamma_m : m \in \mathbb{Z}^d\}} \subseteq T^d$ be a closed subgroup whose nature strongly depends on θ. One can examine what the closed subgroups of $\mathbb{T}^d$ look like, which are usually some lower-dimensional torus times a finite abelian group. Define $Q(a) = \int_H \alpha_s(a)ds$, which is a conditional expectation, with $Q^2 = Q$, $Q(1_A) = 1_A$, also $a > 0 \implies Q(a) > 0$ and $\|Q\| = 1$. Then

$$a \in Z(A_\theta) \iff \delta_m *_\theta a *_\theta \delta_{-m} = a \iff \alpha_{\gamma_m}(a) = a\forall m \in \mathbb{Z}^d \iff \alpha_s(a) = a\forall s \in H \iff Q(a) = a$$

So what we have is the center $Z(A_\theta) = \{a \in A_\theta : Q^\theta(a) = a\}$, i.e. Q is a conditional expectation onto the center of A_θ. Let $a \in Z(A_\theta)$. Then given $\varepsilon >$), we can find $f \in C_C(\mathbb{Z}^d) \subset A_\theta$ with

$\|a-f\|_{A_\theta} < \varepsilon$ (meaning we're just taking the completion). Then $\varepsilon > \|Q(a)-Q(f)\| = \|a-Q(f)\|$. But $Q(f) = \sum f(n)Q(\delta_n)$. If $n \notin D_\theta$, then $\exists m$ so $\gamma_m(n) \neq 1$, so n defines a non-trivial character of H. So $Q(\delta_n) = 0$. Thus $Q(f) \in C_C(D_\theta)$. Thus $a \in \overline{C_C(D_\theta)}^{C^*}$. To sum everything up, we have the following theorem:

Theorem 26.1. $Z(A_\theta) = \overline{C_C(D_\theta)}^{C^*} = \{a \in A_\theta : a_n = 0 \text{ if } n \notin D_\theta\}$

More importantly,

Corollary 26.2. *If $D_\theta = \{0\}$, (i.e. if $e^{2\pi i m\cdot(\theta-\theta^t)n} = 1 \forall n \implies m = 0$ or if $m \cdot (\theta - \theta^t)n \in \mathbb{Z}^d \forall n \implies m = 0$, i.e. if $\gamma(\mathbb{Z}^d)$ is dense in T^d), then $Z(A_\theta) = \mathbb{C}1_{A_\theta}$.*

Now let I be a (closed) ideal in A_θ, then for every $m \in \mathbb{Z}^d$, certainly $\delta_m * I * \delta_{-m} \subseteq I$, i.e. $\alpha_{\gamma_m}(I) \subseteq I \forall m$, i.e. $\alpha_s(I) \subseteq I \forall s \in H$, and so $Q(I) \subseteq I$. So if $I \neq \{0\}$, choose $d \in I$, $d > 0$. So $Q(d) > 0$, where $Q(d) \in Z(A_\theta) \cap I$.

Theorem 26.3. *If $Z(A_\theta) = \mathbb{C}1_{A_\theta}$, then A_θ contains no proper ideal, i.e. A_θ is a simple C^*-algebra.*

27 April 17

Just a few things on T, the circle, α the rotation by $2\pi\theta$, an action of $\mathbb{Z}$. Then looking at $C(T) \times_\alpha \mathbb{Z}$, $C(T)$ is generated as a C^*-algebra by e, $e(t) = e^{2\pi i t}$, unitary. Call its image in $C(T) \times_\alpha \mathbb{Z}$ by U. Let $V \in C(T) \times_\alpha \mathbb{Z}$ be the unitary corresponding to o1 $\in \mathbb{Z}$. Thus $\{U, V\}$ generate $C(T) \times_\alpha \mathbb{Z}$. We also have

$$VUV^* = \alpha_1(U), \qquad \alpha_1(U)(t) = \alpha_1(e) = e(t-\theta) = e(-\theta)e(t) = \bar{e}(\theta)U(t)$$

and $VU = \bar{e}(\theta)UV$, $V^kU = \bar{e}(k\theta)UV^k$, etc. $C(T) \times_{\alpha^\theta} \mathbb{Z}$ is a 2d non-commutative torus. $\theta = \begin{pmatrix} 0 & \theta \\ 0 & 0 \end{pmatrix}$. This is simple for θ irrational.

Vector bundles: Given a topological space M, a vector bundle over M is a topological space B and a surjection $B \xrightarrow{\pi} M$, such that each fiber, $\pi^{-1}(m)$, has a vector space structure, in a *locally* trivial way, i.e. each point m has a neighborhood $\mathcal{O}_m$, such that $\pi^{-1}(\mathcal{O}_m) \cong \mathcal{O}_m \times \mathbb{R}^d$, or $\mathbb{C}^d$. If M is not connected, the d's can vary so the dimension of the fibers vary. Let M be compact. $A = C(M)$. By compactness, there is a finite collection of $\mathcal{O}_{m_j}$'s, which cover M. It's useful on each fiber to have a choice of inner product made in a continuous way. To proceed further, let $\{\varphi_j\}$ be a partition of unity subordinate to the cover $\{\mathcal{O}_j\}$, i.e. $\varphi_j \geq 0$, $\sum \varphi_j = 1$, $\text{supp}(\varphi_j) \leq \mathcal{O}_j$. If M happens to be a smooth manifold, you can certainly phrase everything in smooth terms. For each m, consider the standard inner product on $\mathbb{R}^{d_m}$ or $\mathbb{C}^{d_m}$.

Definition 27.1. ξ is a *continuous cross-section* for $B \to M$ if $\xi : M \to B$, $\pi(\xi(m)) = m$ for all $m \in M$.

Can add continuous sections, multiply them by scalars, but more importantly, you can multiply them by elements of $C(M)$. This works in the non-compact case as well. Mainly if you have some scalar on M, $(f\xi)(m) = f(m) = \xi(m)$. The space of continuous sections is often denoted $\Gamma(B)$.

Thus $\Gamma(B)$ is a $C(M)$-module. (In non-commutative case, this is actually a right module). First pointed out by Swan, but they were antecedents in algebraic geoemtry by Grothendick.

Given $\xi, \eta \in \Gamma(B)$, set

$$\langle \xi, \eta \rangle_A(m) = \sum_j \varphi_j(m) \langle \xi|_{\mathcal{O}_{m_j}}(m), \eta|_{\mathcal{O}_{m_j}}(m) \rangle_j$$

with $C\langle \xi, \eta \rangle_A \in C(M)$. Over $\mathbb{R}$, resp. $\mathbb{C}$, often called a *Riemannian*, resp. *Hermitian*, metric on B. Over $\mathbb{C}$ take the above equality to be linear in second variable, so $\langle \xi, \eta f \rangle_A = \langle \xi, \eta \rangle_A f$, and $\langle \xi f, \eta \rangle_A = f^* \langle \xi, \eta \rangle$, $\langle \xi, \xi \rangle_A \geq 0$, $\langle \xi, \xi \rangle_A = 0 \implies \xi = 0$. Additionally, you need to pay attention to for example, $\Gamma(B) = C([0,1]) = A$, then $\langle \xi, \eta \rangle_A(t) = \underline{t}\bar{\xi}(t)\eta(t)$. $\Gamma(B)$ is self-dual for $\langle \cdot, \cdot \rangle_A$, i.e. if $\varphi \in \mathrm{Hom}_A(\Gamma(B), A_A)$, then there is η_φ so $\varphi(\xi) = \langle \eta_\varphi, \xi \rangle_A$.

Now we certainly have that $A \subseteq \mathrm{End}_A(\Gamma(B)) = E$, which is in general a non-commutative algebra. The 1_A acts as the 1_{End}. Now you can do the following: for $\xi, \eta \in \Gamma(B)$, define $\langle \xi, \eta \rangle_E \in E$ by $\langle \xi, \eta \rangle_E \zeta = \xi \langle \eta, \zeta \rangle_A$ (note that this in now linear in the first variable and conjugate linear in the second). For $T \in E$, $\langle T\xi, \eta \rangle_E = T\langle \xi, \eta \rangle_E$. From the self-duality, T^* exists, and $\langle \xi, T\eta \rangle_E = \langle \xi, \eta \rangle_E T^*$. Set $\|\xi\| = \|\langle \xi, \xi \rangle_A\|_A^{\frac{1}{2}}$ is a norm, and it's straightforward to check that $\Gamma(B)$ is complete for this norm. Thus we have an operator norm on E, and we can show that E is a C^*-algebra.

A lot of this works in more general situations, specifically the non-commutative case. The issue is that if you look at the two properties:

1. $\langle \xi, T\eta \rangle_E = \langle \xi, \eta \rangle_E T^*$
2. $\|\xi\| = \|\langle \xi, \xi \rangle_A\|_A^{\frac{1}{2}}$

then the linear span of $\{\langle \xi, \eta \rangle_E\text{'s}\}$, is a 2-sided ideal in E. To show, in fact is $= E$, with $1_E = \sum^{\text{finite}} \langle \zeta_j, \zeta_j \rangle_E$. This is a property that characterizes finitely generated projective modules.

28 April 19

Note: If you look at the literature of vector bundles, you generally deal with principal bundles, however in the non-commutative case, they seem to disappear, and thus the reason they aren't mentioned here.

From last time, we had a vector bundle B over M, with M compact, $A = C(M)$, $\Gamma(B)$ the module of continuous cross-sections, and $\langle \cdot, \cdot \rangle_A$. Finite open cover $\{\mathcal{O}_j\}$ of M such that $B|_{\mathcal{O}_j} \cong \mathcal{O}_j \times \mathbb{R}^d$ (or $\mathbb{C}^d$). Partition of unity $\{\varphi_j\}$ for $\{\mathcal{O}_j\}$, with $1 \geq \varphi_j \geq 0$. Let $E = \mathrm{End}_A(\Gamma(B))$. Define $\langle \xi, \eta \rangle_E \in E$ with $\langle \xi, \eta \rangle_E \zeta = \xi \langle \eta, \zeta \rangle_A$. Because A is commutative, $A \subseteq E$ in an evident way where $1_A \leftrightarrow 1_E$. For each j, choose $\psi_j \in A$, $\psi_j = \begin{cases} 1 & \text{on support } \varphi_j \\ 0 & \text{outside } \mathcal{O}_j \end{cases}$ Let $\eta_k^j = \psi_j e_k$, $\{e_k\}$ is the standard basis for $\mathbb{R}^{d_j}$. Then $\sum_k \langle \eta_k^j, \eta_k^j \rangle_E \geq \varphi_j 1_E$. Now we have $T := \sum_j \sum_k \langle \eta_k^j, \eta_k^j \rangle_E \geq \sum \varphi_j 1_E = 1_E$. Certainly T must be invertible, and it's positive so it must have a square root. Set $\tilde{\eta}_k^j = T^{-\frac{1}{2}} \eta_k^j$. So $\sum \langle \tilde{\eta}_k^j, \tilde{\eta}_k^j \rangle_E = 1_E$.

Definition 28.1. Let A be a unital C^*-algebra, Ξ a right A-module with $\langle \cdot, \cdot \rangle_A$. By a *standard A-module frame* we mean a finite set $\{\eta_j\}$ of elements of Ξ such that $\sum \langle \eta_j \eta_j \rangle_E = 1_E$, where again $E = \mathrm{End}_A(\Xi)$, i.e. the "reconstruction formula" $\xi = \sum_{j=1}^m \eta_j \langle \eta_j, \xi \rangle_A$ holds for all $\xi \in \Xi$.

You see this sort of thing a lot in the signal processing literature. In case you're worried about errors in your data, you take extra vectors in say $\mathcal{H}$ to get a frame to obtain a redundant system.

Thus for the vector bundle case, $\Gamma(B)$ has a standard module frame. One doesn't really have bases in this case, but the standard frame provides a similar tool. A good question would be to see what's the least number of elements needed in the frame. This definitely depends on how whether you're over the complex or real numbers.

Theorem 28.2. *If Ξ has a standard module frame, $\{\eta_j\}_{j=1}^m$, then Ξ is a finitely generated (in the sense of the reconstruction formula above) projective A-module, i.e. Ξ is isomorphic to a direct summand of the free A-module A^k for some k. (We view this as a right A module in the obvious way). In fact, if you define $P \in M_m(A) = \mathrm{End}_A(A^m)$ by $P_{jk} = \langle \eta_j, \eta_k \rangle_A$, then $P^2 = P = P^*$, and $\Xi \cong PA^m$*

Notice that very little uniqueness for the module frames, and therefore very little uniqueness about the projections you obtain. The converse of the theorem also works. On A^m, we have $\langle \cdot, \cdot \rangle_A$ where $\langle (a_j), (b_j) \rangle_A = \sum a_j^* b_j$ with standard basis $e_j = (0, ..., 1_A, ...0)$. For a projective $P \in M_m(A)$, set $\Xi = PA^m$ and $\eta_j = Pe_j$. Then $\{\eta_j\}$ is a standard module frame for Ξ.

Proof.

$$(P^2)_{ik} = \sum_j P_{ij} P_{jk} = \sum \langle \eta_i, \eta_j \rangle_A \langle \eta_j, \eta_k \rangle_A = \sum \langle \eta_i, \eta_j \langle \eta_j, \eta_k \rangle_A \rangle_A$$

Then $\langle \eta_i, \sum \eta_j \langle \eta_j, \eta_k \rangle_A \rangle_A = \langle \eta_i, \eta_k \rangle = P_{ik}$. So $P^2 = P$ and $(P^*)_{ij} = \langle \eta_j, \eta_i \rangle_A^* = \langle \eta_i, \eta_j \rangle = P_{ij}$ □

29 April 21

30 April 24

M a locally compact Abelian group. We have $M \times \hat{M}$, $L^2(M)$, $\langle \cdot, \cdot \rangle_{L^2}$ linear in the first variable, with

$$(\pi_{x,s}\xi)(y) = \langle y, s \rangle \xi(y - x)$$

and $\pi_u \pi_v = \beta(u, v)\pi_{u+v}$, $\beta((x, s), (y, t)) = \overline{\langle x, t \rangle}$, $\pi_u^* = \beta(u, u)\pi_{-u}$.

Let D be a (discrete) subgroup of $M \times \hat{M}$. Want a right action of $C_C(D)$, $\xi * f = \sum (\pi_u^* \xi) f(u)$. This gives a right representation of $A = C^*(D, \bar{\beta})$. This β is a bi-character, so you certainly can find a matrix β for which it's given by in the case where $D = \mathbb{Z}^n$. We're trying to construct a projective module out of this A from this set up, and we don't want to take all of the L^2 functions so we look inside the functions of compact support. Even nicer would be to take the Schwartz functions, but that requires many more things.

So for $\xi, \eta \in C_C(M) \subseteq L^2(M)$, $u \in D$, set

$$\langle \xi, \eta \rangle_A(u) = \overline{\langle \xi, \pi_u \eta \rangle}_{L^2(M)}$$

Suppose we take $\langle \xi, \eta * f\rangle_A(u) = \sum_{v\in D}\langle \xi, \eta * \delta_v\rangle_A(u) f(v)$, the we need to look at

$$\begin{aligned}
\langle \xi, \eta * \delta_v\rangle_A(u) &= \langle \xi, \pi_v^* \eta\rangle_A(u) \\
&= \langle \xi, \beta(v,v)\pi_{-v}\eta\rangle_A(u) \\
&= \langle \xi, \pi_{-v}\eta\rangle_A(u)\beta(v,v) \\
&= \overline{\langle \xi, \pi_u(\pi_{-v}(\eta))\rangle}_{L^2}\beta(v,v) \\
&= \overline{\langle \xi, \beta(u,-v)\pi_{u-v}\eta\rangle}_{L^2}\beta(u,u) \\
&= \overline{\langle \xi, \pi_{u-v}\eta\rangle}_{L^2}\beta(u,-v)\beta(v,v) \\
&= \langle \xi, \eta\rangle_A(u-v)\beta(v-u,v)
\end{aligned}$$

and we want to check if this equals $\langle \xi, \eta\rangle_A * \delta_v$. We need to then look at

$$\begin{aligned}
(g * \delta_v)(u) &= \sum_w g(w)\delta_v(u-w)\bar{\beta}(w, u-w) \\
&= g(u-v)\bar{\beta}(u-v,v) \\
&= g(u-v)\beta(v-u,v)
\end{aligned}$$

and “by George it works!”

Thus $\langle \xi, \eta * f\rangle_A = \langle \xi, \eta\rangle_A * f$, for $f \in C_C(D) \subset C^*(D,\bar{\beta})$. There's more to check so we should look at $\langle \xi, \eta\rangle_A^*$ and check to see if it's the same as $\langle \eta, \xi\rangle_A$. Then we look at

$$\begin{aligned}
\langle \xi, \eta\rangle_A^*(u) &= \overline{\langle \xi, \eta\rangle_A(-u)}\bar{\beta}(u,u) \\
&= \langle \xi, \pi_{-u}\eta\rangle_{L^2}\bar{\beta}(u,u) \\
&= \langle \pi_{-u}^*\xi, \eta\rangle_{L^2}\bar{\beta}(u,u) \\
&= \langle \pi_u\xi\beta(-u,-u), \eta\rangle_{L^2}\bar{\beta}(u,u) \\
&= \langle \pi_u\xi, \eta\rangle_{L^2} \\
&= \overline{\langle \eta, \pi_u\xi\rangle}_{L^2} \\
&= \langle \eta, \xi, \rangle_A(u)
\end{aligned}$$

Need: for $\xi \in C_C(M)$, have $\langle \xi, \xi\rangle_A \geq 0$, and we can view it in the completed C^*-algebra $C^*(D,\bar{\beta})$.

On $C^*(D,\bar{\beta})$ have the dual group action α of $\hat{D}$. Also, on $C^*(D,\bar{\beta})$ there is a unique α-invariant tracial state, τ, with $\tau(f) = f(0)$. The GNS Hilbert space for τ is $l^2(D)$, with the left regular representation of $C^*(D,\bar{\beta})$ by left multiplication (twisted convolution).

The kernel of this left regular representation ρ is $\{0\}$. Because the action α of $\hat{D}$ on $C^*(D,\bar{\beta})$ is unitarily implemented on $l^2(D)$, $\rho(\alpha_r(a)) = U_r\rho(a)U_r^*$. So the kernel of ρ is an α-invariant ideal by a, (clear from staring). But $C^*(D,\bar{\beta})$ has no proper α-invariant ideal. The case we're really interested in is where $D = \mathbb{Z}^n$, where we know all these facts, but it does work exactly in this generality. Thus we know that ρ is a *faithful* representation of $C^*(D,\bar{\beta})$. So the norm on $C^*(D,\bar{\beta})$ coincides with the norm from the action of $l^2(D)$.

31 April 26

For $\xi, \eta \in C_C(M)$, $A = C^*(D,\bar{\beta})$

$$\langle \xi, \eta\rangle_A(u) = \overline{\langle \xi, \pi_u\eta\rangle}_{L^2(M)}, \quad u \in D, \quad u = (x,s)$$

Why is the this inner product in A? First, this equals $\int_M \bar{\xi}(x)\langle z, s\rangle \eta(x-z)dz$. Looking at Schwartz functions on M, $\mathcal{S}(M)$, $\xi, \eta \in \mathcal{S}(M) \implies \langle \xi, \eta\rangle_A \in \mathcal{S}(D) \subset l^1(D) \subset C^*(D, \bar{\beta}) = A$. Now why is $\langle \xi, \xi\rangle_A \geq 0$ in A? $l^2(D) = L^2(A, \tau)$. For $f \in C_C(D) \subset L^2(A, \tau)$, $\langle\langle \xi, \xi\rangle_A f, f\rangle_{L^2(A,\tau)} \stackrel{\text{def}}{=} \tau(f^*(\langle \xi, \xi\rangle_A f)) = \tau(\langle \xi * f, \underbrace{\xi * f}_{\eta}\rangle_A)$. But for $\eta \in C_C(D)$, $\tau(\langle \eta, \eta\rangle_A) = \langle \eta, \eta\rangle_A(0_D) = \overline{\langle \eta, \eta\rangle}_{L^2(M)} \geq 0$. Now when you see this, you certainly have for $\eta \in C_C(M)$, $\mathcal{S}(M)$, $\langle \eta, \eta\rangle_A \equiv 0$, then $\eta \equiv 0$.

In this setting, define the norm $\|\xi\| = (\|\langle \xi, \xi\rangle_A\|_A)^{\frac{1}{2}}$, and you can now complete $\mathcal{S}(M)$ for $\|\cdot\|$ to get Ξ to be a right module over this completion A and $\langle \cdot, \cdot\rangle_A$ extends to Ξ. Want to know if this is a projective module. It's not with the hypothesis in place so far. We need to use this business of module frames and look at $E = \text{End}_A(\Xi)$. The (finite) linear span of the range of $\langle \cdot, \cdot\rangle_A$ is an ideal in A. For our case, with D discrete, this ideal is all of A, because you easily check that this ideal is invariant under the dual action. We saw earlier that for this class of examples, there are no proper ideals invariant under the dual action.

For $u, v \in M \times \hat{M}$,

$$\pi_u \pi_v = \beta(u, c)\pi_{u+v} = \underbrace{\beta(u, v)\bar{\beta}(v, u)}_{\rho(u,v)} \pi_v \pi_u$$

So $\pi_u\pi_v = \pi_v\pi_u \iff \rho(u, v) = 1$. Let $D^\perp = \{u \in M \times \hat{M} : \rho(u, v) = 1 \forall v \in D\}$. Each π_u for $u \in D^\perp$ is in E. Once you've completed, if you look at the C^*-algebra generated by all of these π_u, $C^*(D^\perp, \beta_{D^\perp}) \subseteq E$ and ask if they're equal. But it's we can see that the left will certainly have an identity element but the right will not. We'll finish this up next time.

32 April 28

So we have $M \times \hat{M}$, D closed subgroup, with $\beta((m, s), (n, t)) = \overline{\langle m, t\rangle}$, $\rho(u, v) = \beta(u, v)\bar{\beta}(v, u) = \rho((m, s), (n, t)) = \langle m, t\rangle\overline{\langle n, s\rangle}$ and you get any character of $M \times \hat{M}$.

$$\begin{aligned} D^\perp &= \{u \in M \times \hat{M} : \rho(u, v) = 1 \forall v \in D\} \\ &= \text{the character on } M \times \hat{M} \text{ which are} \equiv 1 \text{ on } D \\ &= ((M \times \hat{M})/D)\hat{\ } \end{aligned}$$

This is definitely requiring that $\hat{\hat{M}} = M$.

Proposition 32.1. *$D^\perp$ is discrete $\iff$ $(M \times \hat{M})/D$ is compact.*
D is discrete $\iff$ $(M \times \hat{M})/D^\perp$ is compact.
Then $A = C^*(D, \bar{\beta})$ and $C^*(D^\perp, \beta)$ are unital C^*-algebras.

So we want D to be discrete and cocompact, i.e. $(M \times \hat{M})/D$ compact.

Definition 32.2. For G a locally compact group, a *(uniform) lattice* is a discrete subgroup D with G/D compact.

So we want D a lattice, and thus $D^\perp$ is a lattice.

Theorem 32.3. *For D a uniform lattice, $\text{End}_A(\Xi^\cdot) = C^*(D^\perp, \beta)$ and Ξ is a projective A-module. Note that $^\cdot$ means the completion of $\mathcal{S}(M)$.*

In the case $d = 2$, $D \cong \mathbb{Z}^2$. Let $M = \mathbb{R} \times \mathbb{Z}_q$. Already $M = \mathbb{R}$ is interesting. We propose to carry it out in this simple case and then the reader can imagine it for the more complicated case since bookkeeping is the only difference. Now $M \times \hat{M} \cong \mathbb{R} \times \mathbb{R}$, $\langle s, t\rangle = \bar{e}(st)$. $\varphi : \mathbb{Z}^2 \sim D \to \mathbb{R} \times \mathbb{R}$. Given $\theta \in RR$, $\theta \neq 0$, have $\varphi((m, n)) = (-\theta m, n)$. In particular, the action on $C_C(\mathbb{R})$ or $\mathcal{S}(\mathbb{R}) \subset L^2(\mathbb{R})$, gives $(\pi_{(m,n)}\xi)(t) = e(tn)\xi(t - (-\theta m))$. Just an aside, to check things, $\bar{\beta}(\varphi(m, n), \varphi(m', n')) = \bar{\beta}((-\theta m, n), (-\theta m', n')) = e(\theta m, n')$ a cocycle on $\mathbb{Z}^2$. We want a right action: $\xi * \delta_{(m,n)} = (\pi^*_{\underbrace{(-\theta m, n)}_{\in \mathbb{R}\times\hat{\mathbb{R}}}}\xi)(t) = \beta((-m\theta, n), (-\theta m, n))\pi_{(\theta m, -n)}\xi$. So

$$(\xi * \delta_{(m,n)})(t) = e(\theta mn)\bar{e}(tn)\xi(t + \theta m) = \bar{e}((t - \theta m)n)\xi(t - \theta m)$$

with $\xi \in \mathcal{S}(\mathbb{R})$. Then

$$\langle \xi, \eta\rangle_A(m, n) = \overline{\langle \xi, \pi_{\varphi(m,n)}\eta\rangle}_{L^2(\mathbb{R})} = \int \bar{\xi}(t)\bar{e}(tn)\eta(t + \theta m).$$

Note that $\bar{e}((t - \theta m)n)$ is really a 1-periodic function on $\mathbb{R}$, i.e. $\in C(T)$ where $T = \mathbb{R}/\mathbb{Z}$. Then the closed span of $(\xi * \delta_{(m,n)})(t) \sim C(T) \times_{\tau\theta} \mathbb{Z} = A$. $F \in C_C(Z, C(T))$, with

$$(\xi F)(t) = F(\cdot, m)\xi(\cdot))(t - \theta m) = F(t - \theta m, m)\xi(t - \theta m)$$

Now if you have $C_C(G)$ with $A = C^*(H, G/D)$, $B = C^*(K, H/G)$, $H = C(T)$, then defining inner products $\langle\cdot,\cdot\rangle_A$, $\langle\cdot,\cdot\rangle_B$ on $C_C(G)$, you get a Morita equivalence between A and B. This is basically a special case of having a locally compact group X, and H and K groups acting on X, with the actions commuting, and the actions are free and proper.

33 May 3

Let $A = C(\mathbb{R}/\mathbb{Z}) \times_{\tau^\theta} \mathbb{Z}$, $\Xi = C_C(\mathbb{R})$ be a right A-module, $B \subseteq \mathrm{End}_A(\Xi)$ with $B = C(\mathbb{R}/\frac{1}{\theta}\mathbb{Z}) \times_{\tau^1} \mathbb{Z}$ and

$$\langle \xi, \eta\rangle_B(t, u) = \sum_m \xi(t - \theta m)\bar{\eta}(t - \theta m - n)$$

Assume $0 < \theta < 1$.
<u>Claim</u>: $\exists \xi \in \Xi$ such that $\langle \xi, \xi\rangle_B = 1_B$.
So want $\langle \xi, \xi\rangle_B(t, u) = 0$ if $n \neq 0$. So if ξ is supported in an interval of length ¡1, then the above holds. Then we need $\langle \xi, \xi\rangle_B(t, 0) \equiv 1_{C(\ )} = \sum |\xi(t - \theta m)|^2$ with $\xi = \chi_{[0,\theta)}$. Choose small $\varepsilon > 0$, $\varepsilon < 1 - \theta$, $\varepsilon < \theta$. For $-\varepsilon \leq t \leq 1$ we want $|\xi(t)|^2 + |\xi(t + \theta)|^2 = 1$. Then $\langle \xi, \xi\rangle_A$ is a projection, in A, with

$$\begin{aligned} \langle \xi, \xi\rangle_A^2 &= \langle \xi, \xi\rangle_A\langle \xi, \xi\rangle_A & (1)\\ &= \langle \xi, \xi\langle \xi, \xi\rangle_A\rangle_A & (2)\\ &= \langle \xi, \underbrace{\langle \xi, \xi\rangle_B}_{=1}\xi\rangle_A & (3)\\ &= \langle \xi, \xi\rangle_A & (4)\end{aligned}$$

So what is $p = \langle \xi, \xi \rangle_A$? Well,

$$\langle \xi, \eta \rangle_A(s, m) = \sum_n \bar{\xi}(s+n)\eta(s+n+m\theta)$$

Then $p(s, 0) = \langle \xi, \xi \rangle_A(s, 0) = \sum |\xi(s_n)|^2$. For $\theta < \frac{1}{2}$, if $m = 1$, we can certainly periodize $\bar{\xi}(s)\xi(s+\theta)$. The point is that p is supported on $\{-1, 0, 1\}$ in $\mathbb{Z}$. Just a few more points: for $1 \le \theta < 2$, need 2 elements for standard module frame and for $2 \le \theta < 3$, need 3 elements, etc. So if you look at A_θ then it's the same rotation as $A_{\theta+n}$ so the algebra doesn't change. But the Ξ for θ is not $\cong \Xi$ for $\theta + n$.

So we have all this fuzziness with all the choices, but there's a way to systematize it all, which is now taking us in the direction of K-theory. On $A_\theta = A$ have canonical $\mathbb{T}^2$ invariant tracial state: $F \in C_C(\mathbb{Z}, C(\mathbb{R}/\mathbb{Z}))$, $tr(F) = \int_0^1 F(s, 0)ds$. So what's the tracial state on p? $tr(p) = \int_0^1 |\xi(s)|^2 ds = \theta$. This is striking because if θ is irrational, you have the trace of a function which is irrational, something which is a novelty.

Now let A be a unital C^*-algebra. We defined $K_0(A)$ and have the relation

$$S(A) \to C(A) \to K_0(A)$$

and can redo it in terms of projections in the various $M_n(A)$.

Any trace τ on A gives a group homomorphism $K_0(A) \to \mathbb{R}$ for a tracial state, τ, get $\tau([1_A]) = 1 \in \mathbb{R}$. Extend τ to $M_n(A)$, $\tau((a_{ij})) = \sum \tau(a_{jj})$. If $p \in M_n(A)$, with $\begin{pmatrix} p & 0 \\ 0 & 0 \end{pmatrix} \in M_{n+m}(A)$,

$$\tau(p) = \tau\left(\begin{pmatrix} p & 0 \\ 0 & 0 \end{pmatrix}\right).$$

If p and $q \in M_n(A)$ give $\cong$ projective A-modules. $\exists x, y \in M_n(A)$, $p = xy$, $q = yx$, then $\tau(p) = \tau(xy) = \tau(yx) = \tau(q)$. Just to tie things up,

$$\tau\left(\begin{pmatrix} p & 0 \\ 0 & 0 \end{pmatrix}\right) = \tau(p) + \tau(q)$$

This is essentially defining the map $S(A) \xrightarrow{\tau} \mathbb{R}$ and you get naturally homomorphisms from $C(A)$ and $K_0(A)$ to $\mathbb{R}$ as well. More specifically, $C(A)$ is a positive cone in $K_0(A)$ and any trace must take $C(A)$ to $\mathbb{R}^+$. Suppose that for all traces on the algebra, you have this happening. Does it come from a trace? This may be an unsolved problem. There's a lot of people involved in trying to determine to what extent the K_0, K_1, etc. determine a C^*-algebra. Now the K data is certainly expanding, but we always want the positive cone. There has been some success by supposing that your C^*-algebra A is simple. The topology starts to get interesting. Chris Phillips has a theorem that if M is a compact manifold, and α is a diffeomorphism giving a minimal action on M, then $C(M) \times_\alpha \mathbb{Z}$ is a simple C^*-algebra.

Now for $C(\mathbb{R}/\mathbb{Z}) \times_{\tau^\theta} \mathbb{Z}$, you have $C(\mathbb{R}/\mathbb{Z}) \supset C(\mathbb{R}/\frac{1}{m}\mathbb{Z})$. You also have $C(\mathbb{R}/\frac{1}{m}\mathbb{Z}) \times_{\tau^\theta} \mathbb{Z} \supset C(\mathbb{R}/\mathbb{Z}) \times_{\tau^\theta} \mathbb{Z}$ and $C(\mathbb{R}/\frac{1}{m}\mathbb{Z}) \times_{\tau^\theta} \mathbb{Z} \cong C(\mathbb{R}/\mathbb{Z}) \times_{\tau^{m\theta}} \mathbb{Z}$. You can get in $C(\mathbb{R}/\mathbb{Z}) \times_{\tau^{m\theta}} \mathbb{Z}$ a projection p with $tr(p) = \{m\theta\}$, $m > 0$, a fractional part between $0 \le \{m\theta\} < 1$. You also get that $1 - p$ is a projection in A_θ.

34 May 5

Let Ξ be a projective right A-module. $A_\theta = C^*(\mathbb{Z}^2, c_\theta) = C^*(C(T), \mathbb{Z}, \tau_\theta)$, $\Xi = C_C(\mathbb{R}) \sim \mathcal{S}(\mathbb{R})$. Let $G = \mathbb{T}^2 = \hat{\mathbb{Z}}^2$ have a *dual action* on A_θ. One copy of $\mathbb{T}$ acts by translation on $C(T)$, while the other copy by pointwise multiplication by $\langle n, t\rangle$. We can then define A_θ^∞ for α. The Lie algebra $\mathfrak{g}$ of G acts on A_θ^∞ be derivations.

Given an algebra A, and a Lie algebra $\mathfrak{g}$, $\delta : \mathfrak{g} \to \mathrm{Der}(A)$. Chern classes over $\mathbb{C}$ are concerned with how $\mathfrak{g}$ relates to projective modules over A. So given a projective right A-module Ξ, we want $\nabla : \mathfrak{g} \to \mathrm{Lin}(\Xi)$. By Leibniz, for $X \in \mathfrak{g}$, $\nabla_X(\xi a) = (\nabla_X \xi)a + \xi(\delta_X a)$. Then ∇ is called a "connection", or "covariant derivative". Problem: In general, can't get ∇ to be a Lie algebra homomorphism.

But connections always exist. For a free module $\Xi = A^n$, $\nabla_X^0((a_j)) \stackrel{\text{def}}{=} ((\delta_X a_j))$. Then ∇^0 is a Lie algebra homomorphism. Any projective module Ξ is isomorphic to pA^n for p a projection in $M_n(A)$. In general, there is no canonical choice for p. So what you do is for $\xi \in \Xi = pA^n$, with $\xi = (a_j)$, set $\nabla_X \xi = p(\nabla_X^0 \xi)$. This certainly gives you a linear map from your Lie algebra to linear operators on Ξ. We then have

$$\nabla_X(\xi a) = p(\nabla_X^0(\xi a)) = p((\nabla_X^0 \xi)a) + \xi(\delta_X a)) = (\nabla_X \xi)a + \xi(\delta_X a)$$

Often the connection we get here is called the "Grassmann connection" for p.

So now we have a whole lot of connections. So we need to look at the space of connections. If ∇^1 and ∇^2 are connections for Ξ, Then

$$(\nabla_X^1 - \nabla_X^2)(\xi a) = (\nabla_X^1 \xi)a + \xi(\delta_X a) - (\nabla^2 \xi)a - \xi(\delta_X a) = ((\nabla_X^1 - \nabla_X^2)\xi)(a)$$

So $\nabla_X^1 - \nabla_X^2 \in \mathrm{End}_A(\Xi)$ and $\nabla^1 - \nabla^2 : \mathfrak{g} \to \mathrm{End}_A(\Xi)$. If $H : \mathfrak{g} \to \mathrm{End}_A(\Xi)$, then $\nabla^1 + H$ is again a connection. The set of connections is an affine space over $\mathrm{Lin}(\mathfrak{g}, \mathrm{End}_A(\Xi))$.

Given a connection, ∇, on Ξ, its curvature, ω^∇, where $\omega^\nabla(X,Y) \stackrel{\text{def}}{=} [\nabla_X, \nabla_Y] - \nabla_{[X,Y]}$ a skew 2-form on $\mathfrak{g}$, with

$$\begin{aligned}
\omega^\nabla(X,Y)(\xi a) &= \nabla_X(\nabla_Y(\xi a)) - \nabla_Y(\nabla_X(\xi a)) - \nabla_{[X,Y]}(\xi a) \\
&= \nabla_X((\nabla_Y \xi)a + \xi(\delta_X a)) - \nabla_Y((\nabla_X \xi)a + \xi(\delta_X a)) - (\nabla_{[X,Y]}\xi)a - \xi(\delta_{[X,Y]}a) \\
&= (\nabla_X(\nabla_Y \xi))a + (\nabla_Y \xi)(\delta_X a) + (\nabla_X \xi)(\delta_X a) + \xi(\delta_x \delta_Y a) - (\nabla_Y(\nabla_X \xi))a \\
&\quad -(\nabla_X \xi)(\delta_Y a) - (\nabla_Y \xi)(\delta_X a) - \xi(\delta_Y \delta_X a) - (\nabla_{[X,Y]}\xi)a - \xi(\delta_{[X,Y]}a)
\end{aligned}$$

This gives that $([\nabla_X, \nabla_Y]\xi)a - (\nabla_{[X,Y]}\xi)(a) = (\omega^\nabla(X,Y)\xi)a$, so $\omega^\nabla(X,Y) \in \mathrm{End}_A(\Xi)$.

If on A we have a "special" trace τ, we get a trace τ on $M_n(A)$. Then $\tau((a_{ij})) \stackrel{\text{def}}{=} \sum \tau(a_{jj})$. For $\Xi = pM_n(A)$, $\mathrm{End}_A(\Xi) = pM_n(A)p \subset M_n(A)$, so you can restrict τ to $E = \mathrm{End}_A(\Xi)$. It's not quite so special because if you have chosen an A-valued inner product $\langle \cdot, \cdot \rangle$ on Ξ, with $\langle \xi, \eta \rangle_E \zeta = \xi \langle \eta, \zeta \rangle_A$, then

$$\tau(\langle \xi, \eta \rangle_E) = \tau(\langle \eta, \xi \rangle_A)$$

So now the point is that you can form $\tau(\omega_{X,Y}^\nabla \in \wedge^2 \mathfrak{g}$. It's a cycle by taking the boundary operator, so it's a cohomology class, and doesn't depend on the connection.

35 May 8

Let $A_\theta = C(\mathbb{T}^2_\theta)$, $\Xi \sim \mathcal{S}(\mathbb{R})$. Have a *Dual* action of $G = \mathbb{T}^2$ on A_θ by

$$(\alpha_{(r,s)}(F))(t,m) = e(sm)F(t+r,m)$$

G has a Lie algebra $\mathfrak{g} = \mathbb{R}^2$, and α gives the action δ of $\mathfrak{g}$ on $A_\theta^\infty = \mathcal{S}(\mathbb{Z}, C^\infty(\mathbb{T}))$ when we consider the twisted product $C(\mathbb{T}) \times_{\tau_\theta} \mathbb{Z}$. Let X_1, X_2 be *standard basis* for $\mathbb{R}^2$. Then

$$(\delta_{X_1}F)(t,m) = F'(t,m), \quad (\delta_{X_2}F)(t,m) = 2\pi i m F(t,m)$$

We want a connection ∇ on $\Xi^\infty \sim \mathcal{S}(\mathbb{R})$. So we just need ∇_{X_1}, ∇_{X_2}. Then define ∇_{X_1} by $(\nabla_{X_1}\xi)(t) = \xi'(t)$. So we have

$$\begin{aligned}
(\nabla_{X_1}(\xi F))(t) &= \nabla_{X_1}(\sum_m \xi(t-\theta m)F(t-\theta m, m)) \\
&= \sum_m \xi'(t-\theta m)F(t-\theta m,m) + \sum \xi(t-\theta m)F'(t-\theta m, m) \\
&= \sum (\nabla_{X_1}\xi)(t-\theta m)F(t-\theta m,m) + \sum \xi(t-\theta m)(\delta_{X_1}F)(t-\theta m,m) \\
&= (\nabla_{X_1}\xi)F + \xi(\delta_{X_1}F)
\end{aligned}$$

Now define $(\nabla_{X_2}\xi)(t) = \frac{2\pi i t}{\theta}\xi(t)$. Then we have

$$(\nabla_{X_2}(\xi F))(t) = \frac{2\pi i t}{\theta}\sum \xi(t-m\theta)F(t-m\theta,m) \stackrel{?}{=} (\nabla_{X_2}\xi)F + \xi(\delta_{X_2}F)$$

$$((\nabla_{X_2}\xi)F)(t) = \sum(\nabla_{X_2}\xi)(t-m\theta)F(t-m\theta,m) = \sum \frac{2\pi i}{\theta}(t-m\theta)\xi(t-m\theta)F(t-m\theta,m)$$

$$(\xi(\delta_{X_2}F)(t) = \sum \xi(t-m\theta)(\delta_{X_2}F)(t-m\theta,m) = \sum \xi(t-m\theta)2\pi i m F(t-m\theta,m)$$

For curvature,

$$\begin{aligned}
(\omega_{X_1,X_2}\xi)(t) &= \nabla_{X_1}(\nabla_{X_2}\xi))t() - (\nabla_{X_2}(\nabla_{X_1}\xi))(t) - \underbrace{\nabla_{[X_1,X_2]}\xi}_{=0} \\
&= \left(\frac{2\pi i t}{\theta}\xi(t)\right)' - \frac{2\pi i t}{\theta}(\xi'(t)) \\
&= \frac{2\pi i}{\theta}\xi(t) + \frac{2\pi i t}{\theta}\xi'(t) - \frac{2\pi i t}{\theta}\xi'(t)
\end{aligned}$$

So $\omega_{X_1,X_2} = \frac{2\pi i}{\theta} I_\Xi \in \mathrm{End}_A(\Xi)$ implies *constant curvature.*

The first Chern class is: $c_1^\Xi(X_1,X_2) = \frac{1}{2\pi i}\tau_E(\omega_{X_1,X_2}) = 1$. We're saying Chern class here, but if you look at the classical literature, it's not exactly the Chern class, though they are in a generalized sense. Certainly, it belongs to $\wedge\mathfrak{g}' = \Omega^1$.

A general definition for a Lie group G acting on an algebra A, with $\mathfrak{g}$ acting by derivations on A, and projective module Ξ, we find a connection ∇ on Ξ and its curvature $\omega \in A \otimes \wedge^2\mathfrak{g}'$. Note that if $\omega_1, \omega_2 \in A \otimes \wedge^2\mathfrak{g}$, then

$$[(\omega_1 \wedge \omega_2)(X^1,X^2,X^3,X^4) = \frac{1}{4!}\sum_{\sigma\in S_4}(\mathrm{sgn}\ \sigma)\omega_1(X_{\sigma(1)},X_{\sigma(2)})\omega_2(X_{\sigma(3)},X_{\sigma(4)})$$

Now $c_k^\Xi(X_1,...,X_{2k}) = \frac{1}{2\pi i}\tau_E(\omega\wedge\omega\wedge\cdots\wedge\omega(X_1,...,X_{2k}))$.

Theorem 35.1. $\partial\omega_k = 0$. *The cohomology class of ω_k is independent of the choice of connection. This class is precisely $c_k(\Xi)$.*

Recall that $H_k = \mathbb{Z}_k/B_k$ with $\mathbb{Z}_k$ = cycles = $\{\omega \in \Omega^k : \partial\omega = 0\}$ and $B_k = \{\partial\omega : \omega \in \Omega^{k-1}\}$. All this generalizes to A, Ξ, any differential calculus $\Omega = (\bigoplus \Omega_k, \partial)$ over A.