Macdonald polynomials and Hilbert schemes

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LECTURE I

Introduction to Hall-Littlewood and Macdonald polynomials, and the $n!$ and $(n+1)^{(n-1)}$ theorems

Hall-Littlewood polynomials from geometry

A *flag* is a chain of subspaces

$$F_\bullet = (0 \subset F_1 \subset F_2 \subset \cdots \subset F_{n-1} \subset \mathbb{C}^n).$$

Denote the usual basis of $\mathbb{C}^n$ by $\{e_1, \ldots, e_n\}$. The *standard flag* $E_\bullet$ is given by

$$E_i = \langle e_1, \ldots, e_i \rangle.$$

Let

$$\begin{aligned} G &= GL_n(\mathbb{C}) \\ B &= \{\text{upper triangular matrices}\} \subseteq G. \end{aligned}$$

Then G acts transitively on

$$X = \{\text{flags}\},$$

and B is the stabilizer of $E_\bullet$, hence

$$X = G/B.$$

Fix a partition of n,

$$\mu = (\mu_1 \geq \mu_2 \geq \cdots \geq \mu_l),$$

and a unipotent matrix

$$g_\mu \in G$$

with Jordan block sizes μ_i.

Example ($n = 5$):

$$\mu = (3, 2)$$

$$g_\mu = \begin{bmatrix} 1 & 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}.$$

Let

$$X_\mu = \{F_\bullet \in X : g_\mu X = X\}$$

be the set of flags fixed by g_μ. X_μ is a *Springer variety*.

Examples:

$$\begin{aligned} X_{(1^n)} &= X, \\ X_{(n)} &= \{E_\bullet\} \text{ is a point,} \\ X_{(2,1)} &= \text{ is a union of two } \mathbb{P}^1\text{'s meeting at } E_\bullet. \end{aligned}$$

Definition. $R_\mu = H^*(X_\mu, \mathbb{C})$.

Some facts:

1. $H^i(X_\mu) = 0$ for i odd; so R_μ is a commutative, graded $\mathbb{C}$-algebra.
2. $R_{(1^n)} = H^*(X) \cong \mathbb{C}[x_1, \ldots, x_n]/(S_n\text{-invariants})$.
3. $X_\mu \subseteq X$ induces a surjection $R_{(1^n)} \twoheadrightarrow R_\mu$, so $R_\mu = R_{(1^n)}/I_\mu$.
4. The ideal I_μ is S_n-invariant, so S_n acts on R_μ.

Problem. Describe the action of S_n on the graded ring R_μ.

I'll write the **solution** in terms of the *characteristic map*

$$\mathcal{F} : S_n\text{-characters} \to \text{symmetric functions}$$

$$\mathcal{F}V = \sum_{|\mu|=n} \dim(V^{S_\mu})\, m_\mu(\mathbf{z})$$

of Frobenius. Here

$$\begin{array}{rcl} V & & \text{is an } S_n\text{-module} \\ \mu & & \text{is a partition of } n \\ S_\mu & = & S_{\mu_1} \times \cdots \times S_{\mu_l} \subseteq S_n \\ & & \text{is a Young subgroup} \\ m_\mu(\mathbf{z}) & = & (z_1^{\mu_1} \cdots z_l^{\mu_l} + \text{symmetric terms}) \\ & & \text{is a monomial symmetric function.} \end{array}$$

Theorem (Frobenius)**.** The characteristic of the irreducible representation V_λ is the *Schur function*

$$\mathcal{F}V_\lambda = S_\lambda(\mathbf{z}).$$

For a graded S_n-module $V = \bigoplus_d V_d$, define the *Frobenius series*

$$\mathcal{F}V(\mathbf{z};t) = \sum_d \mathcal{F}(V_d)\, t^d.$$

Example ($n = 3$):

degree		$R_{(1^3)}$	$R_{(2,1)}$	$R_{(3)}$
	3	$V_{(1^3)}$		
	2	$V_{(2,1)}$		
	1	$V_{(2,1)}$	$V_{(2,1)}$	
	0	$V_{(3)}$	$V_{(3)}$	$V_{(3)}$

Therefore

$$\begin{aligned}
\mathcal{F}R_{(1^3)} &= S_{(3)} + (t+t^2)S_{(2,1)} + t^3 S_{(1^3)} \\
\mathcal{F}R_{(2,1)} &= S_{(3)} + tS_{(2,1)} \\
\mathcal{F}R_{(3)} &= S_{(3)}
\end{aligned}$$

We work with symmetric functions in infinitely many variables

$$\begin{aligned} \Lambda_{\mathbb{Q}(t)}(\mathbf{z}) &= \mathbb{Q}(t)[z_1, z_2, \ldots]^{S_\infty} \\ &\cong \mathbb{Q}(t)[p_1, p_2, \ldots], \end{aligned}$$

where $p_k = m_{(k)} = z_1^k + z_2^k + \cdots$ are the *Newton power-sums*.

Define $\mathbb{Q}(t)$-algebra automorphism $\varepsilon_t : \Lambda \to \Lambda$

$$\varepsilon_t(p_k) = (1 - t^k)p_k$$

and introduce the notation

$$f[Z(1-t)] \overset{\text{def}}{=} \varepsilon_t(f).$$

As motivation, the inverse of ε_t is

$$f \mapsto f(\mathbf{z}, t\mathbf{z}, t^2\mathbf{z}, \ldots),$$

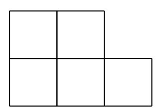
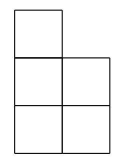
which we might naturally denote by

$$f\left[\frac{Z}{1-t}\right].$$

The *partial ordering* on partitions of n is

$$\lambda \leq \mu \;\Leftrightarrow\; \lambda_1 + \cdots + \lambda_k \leq \mu_1 + \cdots + \mu_k \;\forall k.$$

The *transpose* of a partition is, e.g.,

$\mu = (3, 2) =$  , $\mu' = (2, 2, 1) =$  .

Theorem/Definition. The algebra $\Lambda_{\mathbb{Q}(t)}$ has a basis of *Hall-Littlewood polynomials* $\tilde{H}_\mu(\mathbf{z}; t)$ characterized by

(i) $\tilde{H}_\mu(\mathbf{z}; t) \in \mathbb{Q}(t)\{S_\lambda(\mathbf{z}) : \lambda \geq \mu\}$;

(ii) $\tilde{H}_\mu[Z(1-t); t] \in \mathbb{Q}(t)\{S_\lambda(\mathbf{z}) : \lambda \geq \mu'\}$;

(iii) $\langle \tilde{H}_\lambda, S_{(n)} \rangle = 1$.

Theorem (Hotta–Springer)**.** $\mathcal{F}R_\mu = \tilde{H}_\mu(\mathbf{z}; t)$.

Remark: Define the t-*Kostka coefficients* $\tilde{K}_{\lambda\mu}(t)$ by

$$\tilde{H}_\mu(\mathbf{z};t) = \sum_\lambda \tilde{K}_{\lambda\mu}(t) S_\lambda(\mathbf{z})$$

The Hotta–Springer theorem implies they are non-negative polynomials, $\tilde{K}_{\lambda\mu}(t) \in \mathbb{N}[t]$. *A priori* we only have $\tilde{K}_{\lambda\mu}(t) \in \mathbb{Q}(t)$.

Macdonald polynomials

Now our symmetric functions will involve two parameters, coefficient ring $\mathbb{Q}(q,t)$.

Theorem/Definition (Macdonald)**.** The algebra $\Lambda_{\mathbb{Q}(q,t)}$ has a basis of *Macdonald polynomials* $\tilde{H}_\mu(\mathbf{z};q,t)$ characterized by

(i) $\tilde{H}_\mu[Z(1-q);q,t] \in \mathbb{Q}(t)\{S_\lambda(\mathbf{z}) : \lambda \geq \mu\}$;

(ii) $\tilde{H}_\mu[Z(1-t);q,t] \in \mathbb{Q}(t)\{S_\lambda(\mathbf{z}) : \lambda \geq \mu'\}$;

(iii) $\langle \tilde{H}_\mu, S_{(n)} \rangle = 1$.

Comparing definitions, we see that

$$\tilde{H}_\mu(\mathbf{z}; 0, t) = \tilde{H}_\mu(\mathbf{z}; t).$$

New definition has more symmetry:

$$\tilde{H}_{\mu'}(\mathbf{z}; q, t) = \tilde{H}_\mu(\mathbf{z}; t, q).$$

Define the q, t-*Kostka coefficients* by

$$\tilde{H}_\mu(\mathbf{z}; q, t) = \sum_\lambda \tilde{K}_{\lambda\mu}(q, t) S_\lambda(\mathbf{z}).$$

A priori, $\tilde{K}_{\lambda\mu}(q, t) \in \mathbb{Q}(q, t)$, but. . .

Integrality Theorem (Garsia–Remmel, Garsia–Tesler, Knop, Kirillov–Noumi, Lapointe, Sahi ca. 1995)**.**

$$\tilde{K}_{\lambda\mu}(q, t) \in \mathbb{Z}[q, t].$$

Positivity Theorem (H— 2001)**.**

$$\tilde{K}_{\lambda\mu}(q, t) \in \mathbb{N}[q, t].$$

Macdonald conjectured integrality & positivity in 1988.

An interpretation of $\tilde{H}_\mu(\mathbf{z}; q, t)$

Recall from rep'n theory of S_n that the sign representation $\varepsilon = V_{(1^n)}$ occurs in $V_\lambda \otimes V_\mu$ if and only if $\lambda = \mu'$.

The top degree in R_μ, *i.e.*, $\dim_{\mathbb{C}}(X_\mu)$ is

$$n(\mu) = \sum_i (i-1)\mu_i,$$

and we have

$$(R_\mu)_{n(\mu)} \cong V_\mu$$

V_λ occurs in $(R_\mu)_d$ for $d < n(\mu) \Rightarrow \lambda > \mu$.

Then $R_\mu \otimes R_{\mu'}$ contains ε uniquely, in its top bi-degree $(n(\mu), n(\mu'))$.

Definition. $R_\mu(\mathbf{x}, \mathbf{y}) = R_\mu \otimes R_{\mu'}/J$, where J is the unique largest S_n-invariant ideal not containing ε.

Elementary description—let the boxes in the diagram of μ be $(p_1, q_1), \ldots, (p_n, q_n)$, as shown for $\mu = (3, 2)$:

$(1, 0)$	$(1, 1)$	
$(0, 0)$	$(0, 1)$	$(0, 2)$

In $\mathbb{C}[\mathbf{x}, \mathbf{y}] = \mathbb{C}[x_1, y_1, \ldots, x_n, y_n]$, define the polynomial

$$\Delta_\mu(\mathbf{x}, \mathbf{y}) = \det \begin{bmatrix} x_1^{p_1} y_1^{q_1} & \cdots & x_1^{p_n} y_1^{q_n} \\ \vdots & & \vdots \\ x_n^{p_1} y_n^{q_1} & \cdots & x_n^{p_n} y_n^{q_n} \end{bmatrix}$$

and consider the ideal

$$J_\mu = \{ f \in \mathbb{C}[\mathbf{x}, \mathbf{y}] : f(\frac{\partial}{\partial \mathbf{x}}, \frac{\partial}{\partial \mathbf{y}}) \Delta_\mu = 0 \}.$$

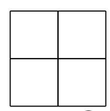
Proposition. $R_\mu(\mathbf{x}, \mathbf{y}) \cong \mathbb{C}[\mathbf{x}, \mathbf{y}] / J_\mu$.

Theorem 1. The Frobenius series of $R_\mu(\mathbf{x}, \mathbf{y})$ as a doubly-graded S_n-module is

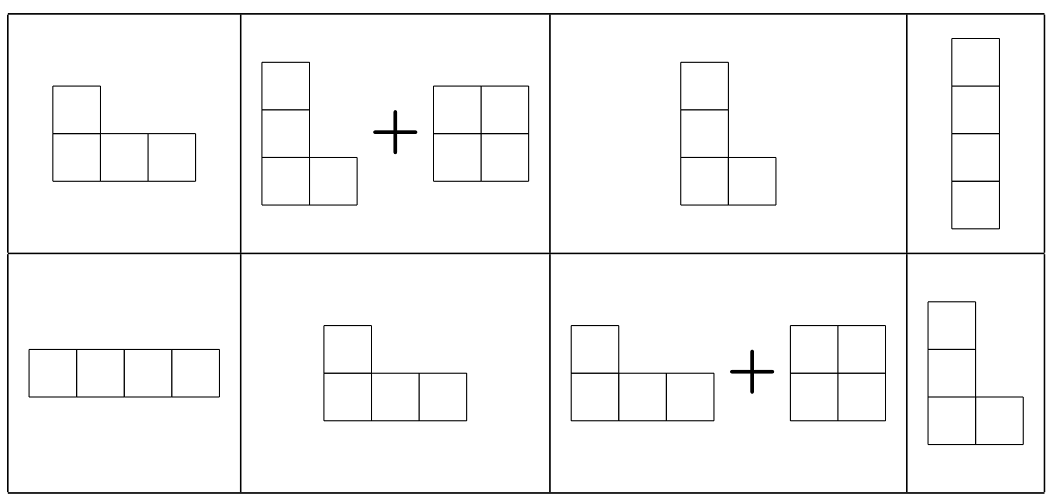
$$\mathcal{F}R_\mu(\mathbf{x}, \mathbf{y}) = \tilde{H}_\mu(\mathbf{z}; q, t).$$

This implies the Positivity Theorem, since

$$\tilde{K}_{\lambda\mu}(q,t) = \sum_{r,s} \mathrm{mult}(V_\lambda, R_\mu(\mathbf{x}, \mathbf{y})_{r,s}) t^r q^s.$$

Example: $R_{(3,1)}(\mathbf{x}, \mathbf{y})$.  stands for $V_{(2,2)}$, so $\tilde{K}_{(2,2),(3,1)}(q,t) = qt + q^2$, and so on.

$\uparrow$ $\mathbf{x}$-degree



$\mathbf{y}$-degree $\rightarrow$

Left column shows Springer ring $R_\mu = R_{(3,1)}$; bottom row is $R_{\mu'} = R_{(2,1,1)}$.

Proposition (Macdonald)**.** Let $V = \mathbb{C}S_n$ be the regular representation. Then $\tilde{H}_\mu(\mathbf{z}; 1, 1) = p_1(\mathbf{z})^n = \mathcal{F}V$, for any μ.

Hence Theorem 1 implies

$$R_\mu(\mathbf{x}, \mathbf{y}) \cong \mathbb{C}S_n$$

as an ungraded S_n-module; in particular,

$$\dim(R_\mu(\mathbf{x}, \mathbf{y})) = n!$$

for every partition μ of n.

For $\mu = (1^n)$, when $R_{(1^n)}(\mathbf{x}, \mathbf{y}) = R_{(1^n)}(\mathbf{x}) = H^*(X)$, this is classical. For general μ, I call it the $\boldsymbol{n!}$ **theorem**. It is equivalent to the existence of certain rank $n!$ vector bundle on the Hilbert scheme (tomorrow's lecture).

The $(n+1)^{n-1}$ theorem

Define the *diagonal coinvariant ring*

$$\begin{aligned} R_n &= \mathbb{C}[\mathbf{x},\mathbf{y}]/(S_n\text{-invariants}) \\ &= \mathbb{C}[\mathbf{x},\mathbf{y}]/(\mathbb{C}[\mathbf{x},\mathbf{y}]_+^{S_n}), \end{aligned}$$

a bivariate analog of the classical coinvariant ring

$$R_{(1^n)} = \mathbb{C}[\mathbf{x}]/(S_n\text{-invariants}) = H^*(X,\mathbb{C}).$$

The rings R_μ in the $n!$ theorem are quotients,

$$R_n \twoheadrightarrow R_\mu.$$

Theorem 2. Let ∇ be the linear operator on $\Lambda_{\mathbb{Q}(q,t)}$ given by

$$\nabla \tilde{H}_\mu(\mathbf{z};q,t) = t^{n(\mu)} q^{n(\mu')} \tilde{H}_\mu(\mathbf{z};q,t),$$

and let $e_n(\mathbf{z})$ be the n-th elementary symmetric function. The Frobenius series of the diagonal coinvariant ring is

$$\mathcal{F}R_n = \nabla e_n(\mathbf{z}).$$

Some remarkable consequences. . .

Corollary 1. We have

$$\dim(R_n) = (n+1)^{n-1},$$

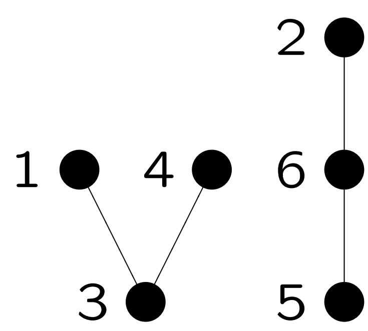
and R_n is isomorphic as an S_n-module to $\varepsilon \otimes V$, where V is the permutation representation of S_n on the finite abelian group

$$(\mathbb{Z}/(n+1)\mathbb{Z})^n / \langle (1,1,\ldots,1) \rangle.$$

Corollary 2. Ignoring the $\mathbf{y}$-grading and considering only $\mathbf{x}$-degree,

$$\dim(R_n)_{d,-}$$

is equal to the number of rooted forests on the vertex set $\{1,\ldots,n\}$ with d inversions [example:



has three inversions: $(1,3)$, $(2,6)$, $(2,5)$].

In Lecture 2, we'll explain these symmetric function formulas by interpreting the rings R_μ and R_n in terms of the Hilbert scheme of points in the plane.

Now consider any Weyl group W, its root lattice Q and defining representation $\mathfrak{h} = Q \otimes_{\mathbb{Z}} \mathbb{C}$.

Theorem (I. Gordon)**.** The diagonal coinvariant ring

$$R_W = \mathcal{O}(\mathfrak{h} \oplus \mathfrak{h})/(W\text{-invariants})$$

has a natural quotient $\widehat{R}_W$ such that

$$\dim(\widehat{R}_W) = (h+1)^r,$$

where h is the Coxeter number and $r = \dim(\mathfrak{h})$ is the rank. Moreover, R_W is isomorphic as a W-module to $\varepsilon \otimes V$, where V is the permutation representation of W on $Q/(h+1)Q$.

Example. For $W = B_4$, $\dim(R_W) = 9^4 + 1$, but $\dim(\widehat{R}_W) = 9^4$. Gordon's method doesn't explain the fact that $R_W = \widehat{R}_W$ for $W = S_n$.

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LECTURE II

The connection between Macdonald polynomials and the Hilbert scheme of points in the plane

Hilbert scheme $H_n = \mathrm{Hilb}^n(\mathbb{C}^2)$

As a *set*. . .

$$\begin{aligned} H_n &= \{\text{finite subschemes } S \subseteq \mathbb{C}^2 \text{ of length } n\} \\ &= \{\text{ideals } I \subseteq \mathbb{C}[x,y] : \dim_{\mathbb{C}}(\mathbb{C}[x,y]/I) = n\} \end{aligned}$$

As a *scheme* (in coordinates). . .

Set $M_\mu = \{x^p y^q : (p,q) \in \mu\}$, e.g. $(n = 5)$

$$M_{(3,2)} = \begin{array}{|c|c|c} \hline x & xy & \\ \hline 1 & y & \multicolumn{1}{c|}{y^2} \\ \hline \end{array}\,.$$

H_n is covered by open affines

$$U_\mu = \{I : M_\mu \text{ spans } \mathbb{C}[x,y]/I\}.$$

Given $(r,s) \notin \mu$, have unique coefficients s.t.

$$x^r y^s \equiv \sum_{(p,q)\in\mu} C^{rs}_{pq}\, x^p y^q \pmod{I}.$$

I ideal $\Leftrightarrow$ certain equations in C^{rs}_{pq}'s hold.

As a *scheme* (functorially)...

Have *tautological family*

$$
\begin{array}{ccc}
F & \subseteq & H_n \times \mathbb{C}^2 \\
\big\downarrow \pi & & \\
H_n & &
\end{array}
$$

with fibers $\pi^{-1}(I) = \mathrm{Spec}(\mathbb{C}[x,y]/I)$.

Universal property: any family

$$
\begin{array}{ccc}
T & \subseteq & Z \times \mathbb{C}^2 \\
\big\downarrow & & \\
Z & &
\end{array}
$$

flat & finite of degree n over Z, is the pullback of F by a unique morphism

$$
\begin{array}{ccc}
T & \longrightarrow & F \\
\big\downarrow & & \big\downarrow \pi \\
Z & \xrightarrow[\phi]{} & H_n\,.
\end{array}
$$

(H_n *represents the functor* of such families.)

More info. . .

Theorem (Fogarty). H_n is non-singular, reduced and irreducible of dimension $2n$.

The *Chow morphism*

$$H_n \xrightarrow[\sigma]{} \mathrm{Sym}^n(\mathbb{C}^2)$$

$$\sigma(S) = \sum_P (\text{length}\, \mathcal{O}_{S,P}) \cdot P.$$

is projective & birational.

Torus $T = (\mathbb{C}^*)^2$ acts on $\mathbb{C}^2$ & H_n. Explicitly,

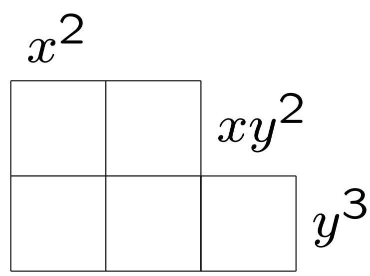
$$(t,q) \cdot I = I|_{x \mapsto t^{-1}x,\, y \mapsto q^{-1}y}.$$

T*-fixed points* of H_n are ideals

$$I_\mu = (x^r y^s : (r,s) \notin \mu),$$

e.g.

$$I_{(3,2)} = (x^2, xy^2, y^3)$$



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G-Hilbert schemes

Let $G =$ finite group acting on $\mathbb{C}^d$. The *Hilbert scheme of regular G-orbits*

$$G\text{-Hilb}(\mathbb{C}^d) = \{G\text{-invariant subschemes } S \subseteq \mathbb{C}^d \text{ such that } \mathcal{O}(S) \underset{G}{\cong} \mathbb{C}G\}$$

is a closed subscheme of $\mathrm{Hilb}^{|G|}(\mathbb{C}^d)$.

Take $G = S_n$ acting on $(\mathbb{C}^2)^n$, with $\mathcal{O}((\mathbb{C}^2)^n) = \mathbb{C}[x_1, y_1, \ldots, x_n, y_n] = \mathbb{C}[\mathbf{x}, \mathbf{y}]$. Let

$$J \subseteq \mathbb{C}[\mathbf{x}, \mathbf{y}], \quad J \in S_n\text{-Hilb}(\mathbb{C}^{2n}).$$

Now x_n, y_n, $x_1^r y_1^s + \cdots + x_{n-1}^r y_{n-1}^s$ generate $\mathbb{C}[\mathbf{x}, \mathbf{y}]^{S_{n-1}}$, and

$$(x_1^r y_1^s + \cdots + x_{n-1}^r y_{n-1}^s) + x_n^r y_n^s \equiv c \pmod{J},$$

hence

$$\mathbb{C}[x_n, y_n] \twoheadrightarrow (\mathbb{C}[\mathbf{x}, \mathbf{y}]/J)^{S_{n-1}}$$

is surjective, with kernel

$$I \subseteq \mathbb{C}[x_n, y_n], \quad I \in H_n.$$

We now have a morhpism

$$\phi\colon S_n\text{-Hilb}(\mathbb{C}^{2n}) \to H_n,$$

which is generically the obvious one:

$$S_n \cdot (a_1, b_1, \ldots, a_n, b_n) \underset{\phi}{\mapsto} \{(a_1, b_1), \ldots, (a_n, b_n)\}.$$

Theorem 1. S_n-Hilb$(\mathbb{C}^{2n}) \cong H_n$.

To prove it, need to construct a family of regular S_n orbits over H_n, so universal property of S_n-Hilb$(\mathbb{C}^{2n})$ will give $\phi^{-1}\colon H_n \to S_n$-Hilb$(\mathbb{C}^{2n})$.

Consider the *reduced* fiber product

$$\begin{array}{ccc} X_n & \longrightarrow & \mathbb{C}^{2n} \\ \rho\big\downarrow & & \big\downarrow \\ H_n & \xrightarrow{\sigma} & \text{Sym}^n(\mathbb{C}^2) = \mathbb{C}^{2n}/S_n. \end{array}$$

Theorem. X_n is Cohen-Macaulay (*i.e.*, ρ is flat) and Gorenstein.

Proof sketch. . . let

$$A = (\mathbb{C}[\mathbf{x},\mathbf{y}]^{\varepsilon})$$

be the ideal in $\mathbb{C}[\mathbf{x},\mathbf{y}]$ generated by antisymmetric polynomials. A description of X_n as a blowup

$$X_n = \mathsf{Proj}(\mathbb{C}[\mathbf{x},\mathbf{y}][tA]),$$

plus a geometric induction on n

$$\begin{tikzcd}
X_{n-1,n} \arrow[rr] \arrow[dr] \arrow[dd] & & X_n \arrow[d] \\
& H_{n-1,n} \arrow[r] \arrow[d] & H_n \\
X_{n-1} \arrow[r] & H_{n-1}, &
\end{tikzcd}$$

reduces us to

Proposition. A^d is a free $\mathbb{C}[\mathbf{x}]$-module for all d.

We prove this by brute force, constructing a basis.

Tying in Lecture 1

Tautological families

$$\begin{array}{ccc} X_n \subseteq H_n \times \mathbb{C}^{2n} & & F \subseteq H_n \times \mathbb{C}^2 \\ \rho \big\downarrow & & \big\downarrow \pi \\ H_n = S_n\text{-Hilb}(\mathbb{C}^{2n}), & & H_n \end{array}$$

give tautological vector bundles $B = \pi_* \mathcal{O}_F$, $P = \rho_* \mathcal{O}_{X_n}$ on H_n, with fibers

$$B(I) = \mathbb{C}[x, y]/I, \quad P(I) = \mathbb{C}[\mathbf{x}, \mathbf{y}]/J,$$

where $J = \phi^{-1}(I)$.

Recall from Lecture 1

$$R_\mu(\mathbf{x}, \mathbf{y}) = \mathbb{C}[\mathbf{x}, \mathbf{y}]/J_\mu,$$

$$J_\mu = \{f \in \mathbb{C}[\mathbf{x}, \mathbf{y}] : f(\frac{\partial}{\partial \mathbf{x}}, \frac{\partial}{\partial \mathbf{y}})\Delta_\mu = 0\}.$$

Proposition. $J_\mu = \phi^{-1}(I_\mu)$, *i.e.*, $R_\mu(\mathbf{x}, \mathbf{y}) = P(I_\mu)$.

Proof: both rings are Gorenstein quotients of $\mathbb{C}[\mathbf{x}, \mathbf{y}]$ with the same socle.

Now recall Macdonald polynomials

(i) $\tilde{H}_\mu[Z(1-q); q, t] \in \mathbb{Q}(t)\{S_\lambda(\mathbf{z}) : \lambda \geq \mu\}$;
(ii) $\tilde{H}_\mu[Z(1-t); q, t] \in \mathbb{Q}(t)\{S_\lambda(\mathbf{z}) : \lambda \geq \mu'\}$;
(iii) $\langle \tilde{H}_\mu, S_{(n)} \rangle = 1$.

Proposition. If $\mathcal{F}V = f(\mathbf{z}, t)$ is the Frobenius series of a graded $S_n * \mathbb{C}[\mathbf{x}]$-module V, then

$$f[Z(1-t)] = \sum_i (-1)^i \mathcal{F}\, \mathrm{Tor}_i^{\mathbb{C}[\mathbf{x}]}(V, \mathbb{C}).$$

Let

$$f_\mu(\mathbf{z}; q, t) = \mathcal{F} R_\mu(\mathbf{x}, \mathbf{y}).$$

Using the Proposition, read off $f_\mu[Z(1-q); q, t]$ and $f_\mu[Z(1-t); q, t]$ from the Koszul homology of $\mathcal{O}_{X_n, \rho^{-1}(I_\mu)}$ w.r.t. $\mathbf{x}$ and $\mathbf{y}$. But $\mathbf{x}$ and $\mathbf{y}$ are regular sequences in $\mathcal{O}_{X_n}$, so this is easy! We verify that f_μ satisfies (i)-(iii) above, hence

$$f_\mu = \tilde{H}_\mu(\mathbf{z}; q, t).$$

Next, the diagonal coinvariants

$$R_n = \mathbb{C}[\mathbf{x}, \mathbf{y}]/(S_n\text{-invariants}).$$

In the diagram

$$\begin{array}{ccc} X_n & \longrightarrow & \mathbb{C}^{2n} \\ {\scriptstyle\rho}\Big\downarrow & & \Big\downarrow{\scriptstyle\psi} \\ H_n & \xrightarrow{\sigma} & \mathrm{Sym}^n(\mathbb{C}^2) = \mathbb{C}^{2n}/S_n, \end{array}$$

Spec(R_n) is the scheme-theoretic fiber $\psi^{-1}(\mathbf{0})$. So $X_n \to \mathbb{C}^{2n}$ induces a map

$$R_n \to H^0(\rho^{-1}\sigma^{-1}(\mathbf{0}), \mathcal{O}) = H^0(\sigma^{-1}(\mathbf{0}), P).$$

Theorem 2. The (scheme-theoretic) zero-fiber $Z_n = \sigma^{-1}(\mathbf{0})$ is reduced & Cohen-Macaulay, and $\mathcal{O}_{Z_n}$ has an explicit $\mathcal{O}_{H_n}$-locally free resolution.

Theorem 3. $H^i(Z_n, P) = 0$ for $i > 0$, and the above map $R_n \to H^0(Z_n, P)$ is an isomorphism.

About proofs. . . for **Theorem 2**, the zero-fiber in the tautological family F turns out to be a a local complete intersection in F, and Z_n is its isomorphic image. **Theorem 3** follows from Theorem 2 plus a general vanishing theorem

Theorem 4. $H^i(H_n, P \otimes B^{\otimes k})$ for $i > 0$.

This in turn follows from Theorem 1, a theorem of Bridgeland–King–Reid, and the "polygraph theorem" (an intermediate result in the proof of Theorem 1).

We can now write down $\mathcal{F}R_n$ using Thomason's generalized Atiyah–Bott–Lefschetz formula.

$$\mathcal{F}R_n = \sum_{|\mu|=n} \frac{(1-q)(1-t)\Pi_\mu(q,t)B_\mu(q,t)\tilde{H}_\mu(\mathbf{z};q,t)}{\prod_{x\in\mu}(1-q^{-a(x)}t^{l(x)+1})(1-q^{a(x)+1}t^{-l(x)})},$$

where. . .

the sum is over partitions μ of n,

$$B_\mu(q,t) = \sum_{(r,s)\in\mu} t^r q^s,$$

$$\Pi_\mu(q,t) = \prod_{\substack{(r,s)\in\mu \\ (r,s)\neq(0,0)}} (1 - t^r q^s),$$

t	qt	
1	q	q^2

and *arm* $a(x)$ and *leg* $l(x)$ of a box $x \in \mu$ are

	l				
	l				
	x	a	a	a	

$a(x) = 3,\ l(x) = 2.$

Numerator factors

$$(1-q)(1-t)\Pi_\mu(q,t)B_\mu(q,t)$$

come from the free resolution of $\mathcal{O}_{Z_n}$;

$$\tilde{H}_\mu(\mathbf{z}; q,t)$$

comes from the fiber $P(I_\mu)$. Denominator factors

$$\prod_{x\in\mu}(1 - q^{-a(x)}t^{l(x)+1})(1 - q^{a(x)+1}t^{-l(x)})$$

come from torus action on $T^*_{I_\mu}H_n$.

Proposition (Garsia–H—). The expansion of the n-th elementary symmetric function $e_n(\mathbf{z})$ in terms of Macdonald polynomials is

$$e_n(\mathbf{z}) = \sum_{|\mu|=n} \frac{t^{-n(\mu)}q^{-n(\mu')}(1-q)(1-t)\Pi_\mu B_\mu \tilde{H}_\mu(\mathbf{z};q,t)}{\prod_{x\in\mu}(1-q^{-a(x)}t^{l(x)+1})(1-q^{a(x)+1}t^{-l(x)})}$$

Hence

$$\mathcal{F}R_n = \nabla e_n(\mathbf{z}), \text{ where } \nabla \tilde{H}_\mu = t^{n(\mu)}q^{n(\mu')}\tilde{H}_\mu.$$

Set $\mathcal{O}(1) = \bigwedge^n B$. The "miraculous" identity in the Proposition reduces to an instance of Atiyah–Bott for

$$\mathcal{F}H^0(Z_n, \mathcal{O}(-1)\otimes P) = \mathcal{F}V_{(1^n)} = e_n(\mathbf{z}),$$

assuming the truth of

Conjecture. $H^i(Z^n, \mathcal{O}(-1)\otimes P) = 0$ for $i > 0$. More generally (since $\mathcal{O}(-1)$ is a summand of P^*), for $i > 0$

$$H^i(H_n, P^*\otimes P\otimes B^{\otimes k}) = 0.$$

A bigger picture

Fix $\Gamma \subseteq \mathrm{SL}_2(\mathbb{C})$ finite. $G = S_n \wr \Gamma$ acts on $\mathbb{C}^{2n}$. Γ $(\neq 1)$ corresponds to a Dynkin diagram of type A, D, or E.

Conjecture. Quiver varieties $\mathcal{M}(\Lambda_0, \nu)$ associated to affine Dynkin diagrams $\widehat{A}$, $\widehat{D}$, $\widehat{E}$ and the basic weight Λ_0 are *moduli spaces of stable* G*-constellations*.

Our **Theorem 1** on the Hilbert scheme is the case $\Gamma = 1$.

Nakajima & Grojnowski constructed a level-(0,1) representation V_{Λ_0} of the quantum double loop algebra $U_q(\widehat{\widehat{\mathfrak{g}}})$ on $\bigoplus_\nu K_0^{\mathbb{C}^*}(\mathcal{M}(\Lambda_0, \nu))$. The Conjecture would supply a basis consisting of distinguished vector bundles. One expects this to be a "canonical basis" of V_{Λ_0} in some suitable sense.

In type $\widehat{A}_{r-1}$, $\Gamma = \mathbb{Z}/r\mathbb{Z}$ is Abelian and commutes with $T = (\mathbb{C}^*)^2$, which acts on $\mathcal{M}(\Lambda_0, \nu)$ with isolated fixed points. The conjecture gives a tautological bundle P of G-constellations on $\mathcal{M}(\Lambda_0, \nu)$. Its fibers $P(I)$ at fixed points $I \in \mathcal{M}(\Lambda_0, \nu)^T$ are doubly graded G-modules. Their characters should be *wreath Macdonald polynomials*

$$\tilde{H}_I \in \mathbb{N}[q, t] \otimes X(G),$$

determined (conjecturally) by an analog of the definition we gave in Lecture 1 for usual Macdonald polynomials. Plenty of computational evidence suggests that wreath Macdonald polynomials do indeed exist and have coefficients in $\mathbb{N}[q, t]$.

Macdonald polynomials and Hilbert schemes

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LECTURE III

New combinatorial developments in Macdonald theory

Combinatorial formula for $\tilde{H}_\mu(\mathbf{z}; q, t)$

Motivation—

$$\tilde{H}_\mu(\mathbf{z}; 1, 1) = p_1(\mathbf{z})^n = (z_1 + z_2 + \cdots)^n$$

for any μ, where $n = |\mu|$. Assign each *filling*

$$\sigma \colon \mu \to \mathbb{Z}_+$$

the weight

$$z^\sigma = \prod_{x \in \mu} z_{\sigma(x)},$$

e.g.

$$\sigma = \begin{array}{|c|c|c|} \hline 2 & 2 & \multicolumn{1}{c}{} \\ \hline 1 & 5 & 3 \\ \hline 3 & 2 & 4 \\ \hline \end{array}, \qquad z^\sigma = z_1 z_2^3 z_3^2 z_4 z_5.$$

Then

$$p_1(\mathbf{z})^n = \sum_{\sigma \colon \mu \to \mathbb{Z}_+} z^\sigma,$$

and we may expect

$$\tilde{H}_\mu(\mathbf{z}; q, t) = \sum_{\sigma\colon\, \mu \to \mathbb{Z}_+} q^? t^? z^\sigma.$$

Definitions. *Descents* and *major index* of σ:

$$\begin{array}{|c|c|c|c} \cline{1-3} & & & \\ \cline{1-3} & b & & \\ \hline & a & & \multicolumn{1}{c|}{} \\ \hline & & & \multicolumn{1}{c|}{} \\ \hline \end{array} \qquad a < b$$

$$\mathrm{maj}(\sigma) = \sum_{x \in \mathrm{Des}(\sigma)} l(x) + 1$$

(recall *arm* $a(x)$ and *leg* $l(x)$

$$\begin{array}{|c|c|c|c|c|c} \cline{1-4} & l & & & & \\ \cline{1-5} & l & & & \multicolumn{1}{c|}{} & \\ \cline{1-5} & x & a & a & \multicolumn{1}{c|}{a} & \\ \hline & & & & & \multicolumn{1}{c|}{} \\ \hline \end{array} \qquad a(x) = 3,\ l(x) = 2).$$

Example.

$$\begin{array}{|c|c|c|} \cline{1-2} 2 & 2 & \multicolumn{1}{c}{} \\ \hline 1 & 5 & 3 \\ \hline 3 & 2 & 4 \\ \hline \end{array} \qquad \mathrm{maj}(\sigma) = 1 + 2 = 3$$

Inversions of σ

$$\begin{array}{|c|c|c|c|c|}\hline \; & \; & \; \\ \cline{1-4} \; & \; & \; & \; \\ \hline \; & b & \; & a & \; \\ \hline \; & \; & \; & \; & \; \\ \hline \end{array} \quad \underline{\text{or}} \quad \begin{array}{|c|c|c|c|c|}\hline \; & \; & \; \\ \cline{1-4} \; & \; & \; & b \\ \hline \; & a & \; & \; & \; \\ \hline \; & \; & \; & \; & \; \\ \hline \end{array} \qquad a < b$$

$$\mathrm{inv}(\sigma) = |\,\mathrm{Inv}(\sigma)| - \sum_{x \in \mathrm{Des}(\sigma)} a(x).$$

We subtracted "forced" inversions

$$\begin{array}{|c|}\hline b \\ \hline a \\ \hline \end{array} \qquad \boxed{c} \qquad a < b \Rightarrow c < b \text{ or } a < c.$$

Example.

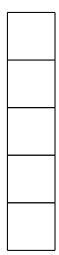
$$\begin{array}{|c|c|c|}\hline 2 & 2 \\ \cline{1-3} 1 & 5 & 3 \\ \hline 3 & 2 & 4 \\ \hline \end{array} \qquad \mathrm{inv}(\sigma) = 5 - 2 = 3$$

Theorem (Haglund–Loehr–H— 2004, conj. by Haglund)**.**

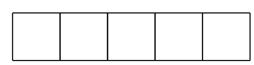
$$\tilde{H}_\mu(\mathbf{z}; q, t) = \sum_{\sigma\colon \mu \to \mathbb{Z}_+} q^{\mathrm{inv}(\sigma)} t^{\mathrm{maj}(\sigma)} z^\sigma.$$

- No combinatorial formula for $\tilde{K}_{\lambda\mu}(q,t)$, as we wrote $\tilde{H}_\mu(\mathbf{z}; q, t)$ in terms of *monomials*, not *Schur functions*.

- Open Problem 1: explain $q \leftrightarrow t$ symmetry $\tilde{H}_{\mu'}(\mathbf{z}; q, t) = \tilde{H}_\mu(\mathbf{z}; t, q)$, generalizing Foata–Schützenberger bijection for $\mu = (1^n)$, (n).



inv $= 0$
maj is
classical



maj $= 0$
inv is classical.

- Open Problem 2: connect combinatorics to R_μ and Hilbert scheme.

- A puzzle: why is our formula a symmetric function in **z**?

LLT polynomials

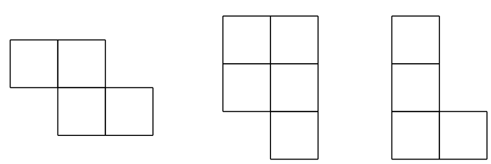
Recall that a *semistandard Young tableau*

$$\begin{array}{|c|c|c|c|c}\hline 4&4&6&7&\multicolumn{1}{c}{}\\\hline 2&3&3&4&\multicolumn{1}{c}{}\\\hline 1&1&1&3&\multicolumn{1}{c|}{5}\\\hline\end{array}$$

is a filling, increasing weakly on rows & strictly on columns. Schur functions are given by

$$S_\lambda(\mathbf{z}) = \sum_{T \in \mathsf{SSYT}(\lambda)} z^T.$$

Fix a tuple $\boldsymbol{\nu}$ of (skew) diagrams

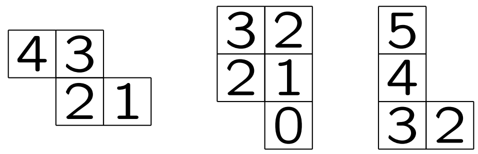


$\nu^{(1)}$ $\nu^{(2)}$ $\nu^{(3)}$

A *semistandard tableau* on $\boldsymbol{\nu}$ is a tuple $T \in \mathsf{SSYT}(\nu^{(1)}) \times \cdots \times \mathsf{SSYT}(\nu^{(k)})$, e.g.

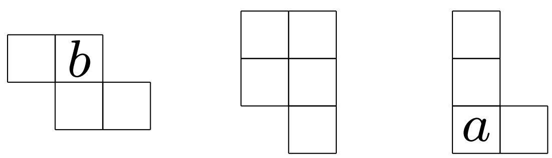
$$\begin{array}{|c|c|c|}\cline{1-2} 2&4&\multicolumn{1}{c}{}\\\hline \multicolumn{1}{c|}{}&3&3\\\cline{2-3}\end{array} \qquad \begin{array}{|c|c|}\hline 2&4\\\hline 1&2\\\hline \multicolumn{1}{c|}{}&1\\\cline{2-2}\end{array} \qquad \begin{array}{|c|c|}\cline{1-1} 5&\multicolumn{1}{c}{}\\\cline{1-1} 3&\multicolumn{1}{c}{}\\\hline 2&3\\\hline\end{array}$$

Mark the *content* $c(x)$ = (row – column) of each box, e.g.



(we may fix a separate origin for each $\nu^{(i)}$).

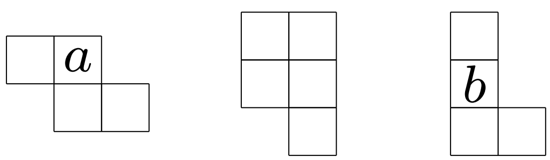
Definition. *Inversions* of SSYT T on $\boldsymbol{\nu}$



$x \in \nu^{(i)}$ $\qquad$ $y \in \nu^{(j)}$

$$a < b, \quad c(y) = c(x), \quad i < j,$$

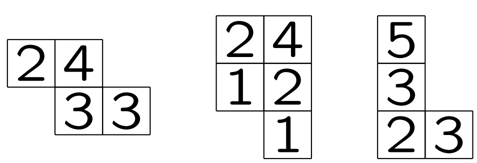
<u>or</u>



$y \in \nu^{(j)}$ $\qquad$ $x \in \nu^{(i)}$

$$a < b, \quad c(y) = c(x) - 1, \quad i > j.$$

Example.

$T =$ 

$$\mathrm{inv}(T) = 10$$

Definition. *LLT polynomial*

$$G_{\boldsymbol{\nu}}(\mathbf{z};q) = \sum_{T \in \mathsf{SSYT}(\boldsymbol{\nu})} q^{|\,\mathsf{Inv}(T)|} z^{T}.$$

Note $G_{\boldsymbol{\nu}}(\mathbf{z};1)$ is a product of skew Schur functions $S_{\nu^{(1)}}(\mathbf{z}) \cdots S_{\nu^{(k)}}(\mathbf{z})$.

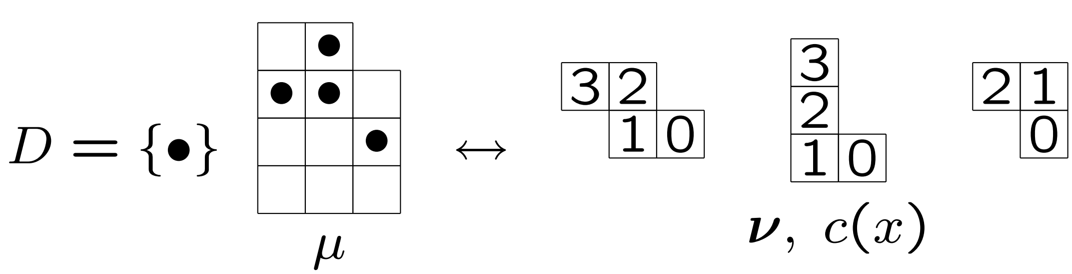
Theorem (Lascoux–Leclerc–Thibon)**.** $G_{\boldsymbol{\nu}}(\mathbf{z};q)$ is a symmetric function.

Proposition. Fix μ and $D \subseteq \mu$. Then

$$F_{\mu,D}(\mathbf{z};q,t) \stackrel{\mathsf{def}}{=} \sum_{\substack{\sigma\colon \mu \to \mathbb{Z}_+ \\ \mathsf{Des}(\sigma) = D}} q^{|\,\mathsf{Inv}(\sigma)|} z^{\sigma} = G_{\boldsymbol{\nu}}(\mathbf{z};q)$$

for a suitable tuple of ribbon skew diagrams $\boldsymbol{\nu}$.

Picture proof.



Corollary. The combinatorial expression for $\tilde{H}_\mu(\mathbf{z}; q, t)$ is a symmetric function, equal to

$$\sum_D t^{\sum_{x\in D}(l(x)+1)} q^{-\sum_{x\in D} a(x)} F_{\mu,D}(\mathbf{z}; q, t).$$

About the proof of our formula... let

$$C_\mu(\mathbf{z}; q, t) = \sum_{\sigma\colon \mu \to \mathbb{Z}_+} q^{\mathsf{inv}(\sigma)} t^{\mathsf{maj}(\sigma)} z^\sigma$$

be the combinatorial expression. *Given that C_μ is symmetric*, we can make sense of

$$C_\mu[Z(1-q); q, t],\ C_\mu[Z(1-t); q, t].$$

We construct sign-reversing involutions to verify C_μ satisfies (i)–(ii) in the def'n of Macdonald polynomials:

(i) $C_\mu[Z(1-q); q, t] \in \mathbb{Q}(t)\{S_\lambda(\mathbf{z}) : \lambda \geq \mu\}$;
(ii) $C_\mu[Z(1-t); q, t] \in \mathbb{Q}(t)\{S_\lambda(\mathbf{z}) : \lambda \geq \mu'\}$;
(iii) $\langle C_\mu, S_{(n)}\rangle = 1$.

For (iii), $\langle C_\mu, S_{(n)}\rangle$ is the coefficient of z_1^n in C_μ. The all-1's filling σ has $\mathsf{maj}(\sigma) = \mathsf{inv}(\sigma) = 0$.

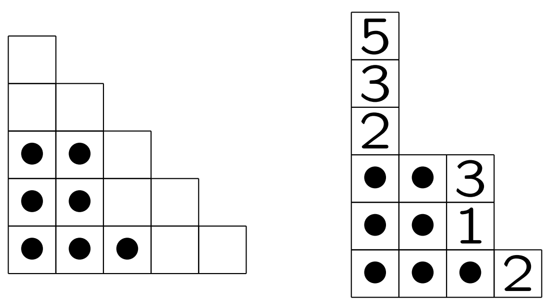
Diagonal coinvariants

Recall

$$R_n = \mathbb{C}[\mathbf{x}, \mathbf{y}]/(S_n\text{-invariants}) = H^0(Z_n, P)$$

$$\mathcal{F} R_n = \nabla e_n(\mathbf{z}).$$

Let $\delta_n = (n-1, n-2, \ldots, 1)$. Consider partitions $\lambda \subseteq \delta_n$ and tableaux $T \in SSYT(\lambda + (1^n)/\lambda)$, e.g. $(n = 6)$

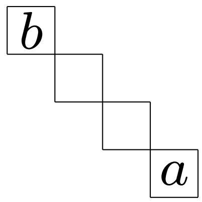


$\lambda = (3, 2, 2)$ $\qquad$ T

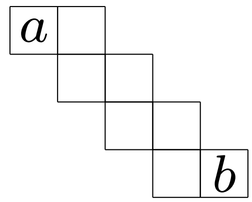
Theorem (Garsia–H—).

$$(\nabla e_n)(\mathbf{z}; 1, t) = \sum_{\lambda \subseteq \delta_n} t^{|\delta_n/\lambda|} S_{\lambda+(1^n)/\lambda}(\mathbf{z}).$$

Definition. *Inversions* of $T \in \mathsf{SSYT}(\lambda+(1^n)/\lambda)$



or

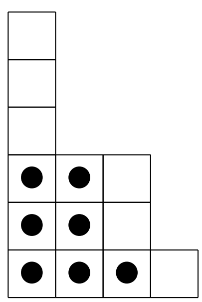


$a < b$

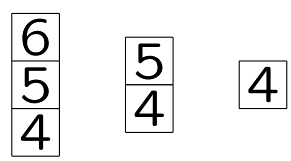
Then

$$\sum_{\substack{T\in\\ \mathsf{SSYT}(\lambda+(1^n)/\lambda)}} q^{|\,\mathsf{Inv}(T)|} z^T$$

is an LLT polynomial (hence symmetric), e.g.



$\leftrightarrow$



$\lambda + (1^n)/\lambda$ $\qquad$ $\boldsymbol{\nu},\ c(x)$

Conjecture (Haglund–Loehr–Remmel–Ulyanov–H—).

$$\nabla e_n(\mathbf{z}) = \sum_{\lambda\subseteq\delta_n} t^{|\delta_n/\lambda|} \sum_{\substack{T\in\\ \mathsf{SSYT}(\lambda+(1^n)/\lambda)}} q^{|\,\mathsf{Inv}(T)|} z^t.$$

- Open Problem 3. Prove the conjecture.
- Open Problem 4. Relate it to geometry.
- Open Problem 5. Exhibit the $q \leftrightarrow t$ symmetry of $\nabla e_n(\mathbf{z})$ in the combinatorial formula.
- Open Problem 6. Find a combinatorial expression for the doubly-graded character of Gordon's module $\widehat{R}_W$ for other Weyl groups W.