SOLUTION: ASSIGNMENT 5

4.1.3 *Is the subset of P_2 given below a subspace? Find a basis if it is.*

$$\{p(t) : p'(1) = p(2)\}$$

(p' is the derivative).

Solution. Let $S = \{p(t) : p'(1) = p(2)\}$. For any $p_1, p_2 \in S$, $k_1, k_2 \in \mathbb{R}$, by

$$(k_1 \cdot p_1 + k_2 \cdot p_2)'|_{t=1} = k_1 p_1'(1) + k_2 p_2'(1) = k_1 p_1(2) + k_2 p_2(2) = (k_1 \cdot p_1 + k_2 \cdot p_2)|_{t=2}$$

we checked $k_1 \cdot p_1 + k_2 \cdot p_2 \in S$. Therefore S is a subspace.

For $p(t) = a + bt + ct^2 \in S$,

$$p'(1) = p(2) \Rightarrow b + 2c = a + 2b + 4c \Rightarrow a + b + 2c = 0 \Rightarrow (a, b, c) = (-\lambda - 2\mu, \lambda, \mu)$$

where $\lambda, \mu \in \mathbb{R}$. Then $p(t) = -\lambda - 2\mu + \lambda t + \mu t^2 = \lambda(t - 1) + \mu(t^2 - 2)$, i.e. $S = \operatorname{span}\{t - 1, t^2 - 2\}$. But $t - 1, t^2 - 2 \in S$ are obviously independent. Therefore $(t - 1, t^2 - 2)$ is a basis of S.

4.1.4 *Required as in 4.1.3*

$$\{p(t) : \int_0^1 p(t)\mathrm{d}t = 0\}$$

Solution. Let $S = \{p(t) : \int_0^1 p(t)dt = 0\}$. For any $p_1, p_2 \in S$, $k_1, k_2 \in \mathbb{R}$, by

$$\int_0^1 (k_1 \cdot p_1 + k_2 \cdot p_2)\mathrm{d}t = k_1 \int_0^1 p_1(t)\mathrm{d}t + k_2 \int_0^1 p_2(t)\mathrm{d}t = k_1 0 + k_2 0 = 0$$

we checked $k_1 \cdot p_1 + k_2 \cdot p_2 \in S$. Therefore S is a subspace.

For $p(t) = a + bt + ct^2 \in S$,

$$\int_0^1 p(t)\mathrm{d}t = 0 \Rightarrow a + \frac{b}{2} + \frac{c}{3} = 0 \Rightarrow (a, b, c) = (-\frac{\lambda}{2} - \frac{\mu}{3}, \lambda, \mu)$$

where $\lambda, \mu \in \mathbb{R}$. Then $p(t) = -\frac{\lambda}{2} - \frac{\mu}{3} + \lambda t + \mu t^2 = \lambda(t - \frac{1}{2}) + \mu(t^2 - \frac{1}{3})$, i.e. $S = \operatorname{span}\{t - \frac{1}{2}, t^2 - \frac{1}{3}$. But $t - \frac{1}{2}, t - \frac{1}{3} \in S$ are obviously independent. Therefore $(t - \frac{1}{2}, t - \frac{1}{3})$ is a basis of S.

4.1.21 *Find a basis for the space of all diagonal* 2×2 *matrices, and determine its dimension.*

Solution. Any diagonal 2×2 matrix looks like

$$\begin{pmatrix} a & 0 \\ 0 & d \end{pmatrix} = a \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} + d \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$$

This tells us that $(\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix})$ is a basis because these two matrices are already independent as in $\mathbb{R}^{2\times 2}$. The dimension is 2.

4.1.22 *Find a basis for the space of all diagonal $n \times n$ matrices, and determine its dimension.*

Solution. Any diagonal $n \times n$ matrix looks like

$$\begin{pmatrix} a_1 & & \\ & \ddots & \\ & & a_n \end{pmatrix} = a_1 E_{11} + \cdots + a_n E_{nn}$$

where E_{ii} is the matrix with entries all 0 except a 1 at the i'th diagonal entry. This tells us that $(E_{11}, \cdots, E_{nn})$ is a basis because these n matrices are already independent as in $\mathbb{R}^{n \times n}$. The dimension is n.

4.1.24 *Find a basis of the space of all upper triangular* 3×3 *matrices and determine its dimension.*

Solution.Any upper triangular 3×3 matrix looks like

$$\begin{pmatrix} a_{11} & a_{12} & a_{13} \\ & a_{22} & a_{23} \\ & & a_{33} \end{pmatrix} = a_{11}E_{11} + a_{12}E_{12} + a_{13}E_{13} + a_{22}E_{22} + a_{23}E_{23} + a_{33}E_{33}$$

where E_{ij} denotes the matrix with entries all 0 except for a 1 at the (i, j)-th place. This tells us that $(E_{11}, E_{12}, E_{13}, E_{22}, E_{23}, E_{33})$ is a basis because these 6 matrices are already independent as in $\mathbb{R}^{3 \times 3}$. The dimension is 6.

4.1.32 *Find a basis of the space of all* 2×2 *matrices* S *such that*

$$\begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix} S = S \begin{pmatrix} 2 & 0 \\ 0 & 0 \end{pmatrix}$$

Solution. Let $S = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$, the condition becomes

$$\begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}\begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} a & b \\ c & d \end{pmatrix}\begin{pmatrix} 2 & 0 \\ 0 & 0 \end{pmatrix}$$

which is simplified to be

$$\begin{pmatrix} a+c & b+d \\ a+c & b+d \end{pmatrix} = \begin{pmatrix} 2a & 0 \\ 2c & 0 \end{pmatrix}$$

By comparing the entries we get

$$\begin{cases} a+c &= 2a \\ b+d &= 0 \\ a+c &= 2c \\ b+d &= 0 \end{cases}$$

so $(a, b, c, d) = (\lambda, \mu, \lambda, -\mu)$. Therefore $S = \begin{pmatrix} \lambda & \mu \\ \lambda & -\mu \end{pmatrix} = \lambda \begin{pmatrix} 1 & 0 \\ 1 & 0 \end{pmatrix} + \mu \begin{pmatrix} 0 & 1 \\ 0 & -1 \end{pmatrix}$, i.e. $\left(\begin{pmatrix} 1 & 0 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 0 & -1 \end{pmatrix}\right)$ is a basis of the space of all S's. The dimension is 2.

4.1.50 *Find all solutions of the differential equation* $f''(x) + 8f'(x) - 20f(x) = 0$.

Solution. Solving the characteristic equation $\lambda^2 + 8\lambda - 20 = 0$ we have two eigenvalues $\lambda_1 = 2, \lambda_2 = -10$. Therefore all solutions are $f(x) = c_1 e^{2t} + c_2 e^{-10t}$, where $c_1, c_2 \in \mathbb{R}$.

Remark. Review how to solve a homogeneous linear ODE in Math 1B. Another way is to follow what we did in Example 18 of Section 4.1.

4.1.58 *In this exercise we will show that the function* $\cos x$ *and* $\sin x$ *span the solution space* V *of the differential equation* $f''(x) = -f(x)$.

a. Show that if $g(x)$ is in V, then the function $g(x)^2 + g'(x)^2$ is constant.

b. Show that if $g(x)$ is in V, with $g(0) = g'(0) = 0$, then $g(x) = 0$ for all x.

c. If $f(x)$ is in V, then $g(x) = f(x) - f(0)\cos x - f'(0)\sin x$ is in V as well. Show then $f(x) = f(0)\cos x - f'(0)\sin x$.

Solution. a. Taking derivative we have

$$(g^2 + (g')^2)' = (g^2)' + ((g')^2)' = 2gg' + 2g'g'' = 2g(g + g'')$$

But $g \in V$ means $g'' = -g$, so the right handed side is zero. The derivative being always zero means $g^2 + (g')^2$ is constant.

b. Because $g(x)^2 + g'(x)^2 = C$ is constant, plugging in $g(0) = g'(0) = 0$ we know the constant $C = 0$. Thus $g(x)^2 + g'(x)^2 = 0$, and since the summands are nonnegative, they must both be zero. In particular, $g(x) = 0$.

c. $g(x)$ is in V because it's a linear combination of f, $\cos x$, $\sin x$ which are all in V. But $g(0) = f(0) - f(0)\cos 0 - f'(0)\sin 0 = 0$, $g'(0) = f'(0) + f(0)\sin 0 - f'(0)\cos 0 = 0$. By b. we know $g(x) = 0$, i.e. $f(x) = f(0)\cos x + f'(0)\sin x$.

4.2.6 *Find out if the transformation it is linear, and when linear, if it is isomorphism.*

$$T(M) = \begin{pmatrix} 1 & 2 \\ 3 & 6 \end{pmatrix}$$

from $\mathbb{R}^{2\times 2}$ to $\mathbb{R}^{2\times 2}$.

Solution. Denote $\begin{pmatrix} 1 & 2 \\ 3 & 6 \end{pmatrix}$ as A. For any $M_1, M_2 \in \mathbb{R}^{2\times 2}$ and $k_1, k_2 \in \mathbb{R}$,

$$T(k_1M_1 + k_2M_2) = (k_1M_1 + k_2M_2)A = k_1M_1A + k_2M_2A = k_1T(M_1) + k_2T(M_2)$$

Therefore T is linear. T is not an isomorphism because A is not invertible. In this case T maps any matrix to a non-invertible matrix so $\operatorname{im}T$ cannot be $\mathbb{R}^{2\times 2}$ (e.g. it avoids all invertible matrices).

4.2.12 *Required as in 4.2.6, $T(c) = c\begin{pmatrix} 2 & 3 \\ 4 & 5 \end{pmatrix}$ from $\mathbb{R}$ to $\mathbb{R}^{2\times 2}$.*

Solution. Denote $\begin{pmatrix} 2 & 3 \\ 4 & 5 \end{pmatrix}$ as A. For any $c_1, c_2 \in \mathbb{R}$ and $k_1, k_2 \in \mathbb{R}$,

$$T(k_1c_1 + k_2c_2) = (k_1c_1 + k_2c_2)A = k_1c_1A + k_2c_2A = k_1T(c_1) + k_2T(c_2)$$

Therefore T is linear. T cannot an isomorphism because the dimensions of target space and the source space are different.

4.2.14 *Required as in 4.2.6, $T(M) = \begin{pmatrix} 2 & 3 \\ 5 & 7 \end{pmatrix}M - M\begin{pmatrix} 2 & 3 \\ 5 & 7 \end{pmatrix}$ from $\mathbb{R}^{2\times 2}$ to $\mathbb{R}^{2\times 2}$.*

Solution. Denote $\begin{pmatrix} 2 & 3 \\ 5 & 7 \end{pmatrix}$ as A. For any $M_1, M_2 \in \mathbb{R}^{2\times 2}$ and $k_1, k_2 \in \mathbb{R}$,

$$T(k_1M_1+k_2M_2) = A(k_1M_1+k_2M_2)-(k_1M_1+k_2M_2)A = k_1(AM_1-M_1A)+k_2(AM_2-M_2A) = k_1T(M_1)+k_2T(M_2)$$

Therefore T is linear. Observe that $T(I_2) = AI_2 - I_2A = A - A = 0$ where I_2 is the unit matrix, and hence T is not an isomorphism because $\ker T \neq \{0\}$ (e.g. it contains I_2).

4.2.26 *Required as in 4.2.6, $T(f(t)) = f(-t)$ from P_2 to P_2.*

Solution. For any $f_1, f_2 \in P_2$ and $k_1, k_2 \in \mathbb{R}$,

$$T(k_1f_1 + k_2f_2)(t) = (k_1f_1 + k_2f_2)(-t) = k_1f_1(-t) + k_2f_2(-t) = k_1T(f_1)(t) + k_2T(f_2)(t)$$

Therefore T is linear. Observe that T^2 is identity, i.e. $T(T(f(t))) = T(f(-t)) = f(t)$, so T is the inverse of itself. T is invertible and hence is isomorphism.

4.2.30 *Required as in 4.2.6, $T(f(t)) = tf'(t)$ from P_2 to P_2.*

Solution. For any $f_1, f_2 \in P_2$ and $k_1, k_2 \in \mathbb{R}$,

$$T(k_1f_1 + k_2f_2)(t) = t(k_1f_1(t) + k_2f_2(t))' = k_1tf_1'(t) + k_2tf_2'(t) = k_1T(f_1)(t) + k_2T(f_2)(t)$$

Therefore T is linear. Observe that $T(c) = t0 = 0$ where c is any constant polynomial, and hence T is not an isomorphism because $\ker T \neq \{0\}$ (e.g. it contains constant polynomials).

4.2.48 *Required as in 4.2.6, $T(f(t)) = f'(t)$ from P to P.*

Solution. For any $f_1, f_2 \in P$ and $k_1, k_2 \in \mathbb{R}$,

$$T(k_1f_1 + k_2f_2)(t) = (k_1f_1(t) + k_2f_2(t))' = k_1f_1'(t) + k_2f_2'(t) = k_1T(f_1)(t) + k_2T(f_2)(t)$$

Therefore T is linear. Observe that $T(c) = t0 = 0$ where c is any constant polynomial, and hence T is not an isomorphism because $\ker T \neq \{0\}$ (e.g. it contains constant polynomials).

Remark. But in this case, $\mathrm{im}T = P$, i.e. it is surjective. The explanation is that P is infinite dimensional.

4.2.56 *Find image,rank, kernel and nullity of the transformation in 4.2.30*

Solution. Let $f(t) = a + bt + ct^2$, then $T(f)(t) = t(a + bt + ct^2)' = bt + 2ct^2$. Thus $\mathrm{im}T = \mathrm{span}\{t, 2t^2\}$. Consider $bt + 2ct^2 = 0$ as a polynomial, i.e. all coefficients are zero, so $(a, b, c) = (\lambda, 0, 0)$ where $\lambda \in \mathbb{R}$. Thus $\ker T = \mathrm{span}\{1\}$. The rank is $\dim \mathrm{im}T = 2$, and the nullity is $\dim \ker T = 1$.

4.2.67 *For which constants k is the linear transformation*

$$T(M) = \begin{pmatrix} 2 & 3 \\ 0 & 4 \end{pmatrix} M - M \begin{pmatrix} 3 & 0 \\ 0 & k \end{pmatrix}$$

an isomorphism from $\mathbb{R}^{2\times2}$ to $\mathbb{R}^{2\times2}$?

Solution. The most standard way to understand a linear transformation is to find out its matrix under a favorable basis. For simplicity we write the standard basis of $\mathbb{R}^{2\times2}$, namely $\left(\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}\right)$ as $(\vec{e}_1, \vec{e}_2, \vec{e}_3, \vec{e}_4)$. Then

$$\begin{aligned} T(\vec{e}_1) &= \begin{pmatrix} 2 & 3 \\ 0 & 4 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} - \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}\begin{pmatrix} 3 & 0 \\ 0 & k \end{pmatrix} \\ &= \begin{pmatrix} -1 & 0 \\ 0 & 0 \end{pmatrix} \\ &= -\vec{e}_1 \end{aligned}$$

Similarly, $T(\vec{e}_2) = (2 - k)\vec{e}_2, T(\vec{e}_3) = 3\vec{e}_1 + \vec{e}_3, T(\vec{e}_4) = 3\vec{e}_2 + (4 - k)\vec{e}_4$. Therefore,

$$\begin{aligned}
T(\begin{pmatrix} a & b \\ c & d \end{pmatrix}) &= T(a\vec{e}_1 + b\vec{e}_2 + c\vec{e}_3 + d\vec{e}_4) \\
&= T\begin{pmatrix} \vec{e}_1 & \vec{e}_2 & \vec{e}_3 & \vec{e}_4 \end{pmatrix}\begin{pmatrix} a \\ b \\ c \\ d \end{pmatrix} \\
&= \begin{pmatrix} T(\vec{e}_1) & T(\vec{e}_2) & T(\vec{e}_3) & T(\vec{e}_4) \end{pmatrix}\begin{pmatrix} a \\ b \\ c \\ d \end{pmatrix} \\
&= \begin{pmatrix} -\vec{e}_1 & (2-k)\vec{e}_2 & 3\vec{e}_1 + e_3 & 3\vec{e}_2 + (4-k)\vec{e}_4 \end{pmatrix}\begin{pmatrix} a \\ b \\ c \\ d \end{pmatrix} \\
&= \begin{pmatrix} \vec{e}_1 & \vec{e}_2 & \vec{e}_3 & \vec{e}_4 \end{pmatrix}\begin{pmatrix} -1 & 0 & 3 & 0 \\ 0 & 2-k & 0 & 3 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 4-k \end{pmatrix}\begin{pmatrix} a \\ b \\ c \\ d \end{pmatrix}
\end{aligned}$$

The matrix here is the matrix of T under the standard basis. Clearly it is an isomorphism if and only if k is neither 2 nor 4.

Remark. The third equality of the last computation formally looks like the associativity, which turns out to be *true* for linear transformation. Anyway, this way of computation helps us a lot to get a correct matrix of a transformation under a given basis. In fact, we may just formally compute $T(\vec{e}_1\ \vec{e}_2\ \vec{e}_3\ \vec{e}_4) = (T(\vec{e}_1)\ T(\vec{e}_2)\ T(\vec{e}_3)\ T(\vec{e}_4)) = (\vec{e}_1\ \vec{e}_2\ \vec{e}_3\ \vec{e}_4)A$ to get the matrix A of T under any basis $(\vec{e}_1\ \vec{e}_2\ \vec{e}_3\ \vec{e}_4)$ without writing down the coordinate column vector (cf. the solution of 4.2.68 as an example).

4.2.68 *For which constants k is the linear transformation*

$$T(M) = M\begin{pmatrix} 5 & 0 \\ 0 & 1 \end{pmatrix} - \begin{pmatrix} 2 & 0 \\ 0 & k \end{pmatrix}M$$

an isomorphism from $\mathbb{R}^{2\times 2}$ *to* $\mathbb{R}^{2\times 2}$*?*

Solution. Use the same method as in 4.2.67, we will find $T(\vec{e}_1) = 3\vec{e}_1, T(\vec{e}_2) = -\vec{e}_2, T(\vec{e}_3) = (5-k)\vec{e}_3, T(\vec{e}_4) = (1-k)\vec{e}_4$. Therefore,

$$\begin{aligned}
& T\begin{pmatrix} \vec{e}_1 & \vec{e}_2 & \vec{e}_3 & \vec{e}_4 \end{pmatrix} \\
= & \begin{pmatrix} T(\vec{e}_1) & T(\vec{e}_2) & T(\vec{e}_3) & T(\vec{e}_4)) \end{pmatrix} \\
= & \begin{pmatrix} 3\vec{e}_1 & -\vec{e}_2 & (5-k)\vec{e}_3 & (1-k)\vec{e}_4 \end{pmatrix} \\
= & \begin{pmatrix} \vec{e}_1 & \vec{e}_2 & \vec{e}_3 & \vec{e}_4 \end{pmatrix}\begin{pmatrix} 3 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 5-k & 0 \\ 0 & 0 & 0 & 1-k \end{pmatrix}
\end{aligned}$$

The matrix of T is invertible if and only if k is neither 5 nor 1.