Solutions: Assignment 10

7.2.2 The eigenvalues are 2 and 1, each with multiplicity 2.

7.2.8 The polynomial is $(\lambda+3)\lambda^2$, so the eigenvalues are 0 with multiplicity 2 and -3 with multiplicity 1.

7.2.14

$$det\left(\lambda I - \begin{bmatrix} B & C \\ 0 & D \end{bmatrix}\right) = det\left(\begin{bmatrix} \lambda I - B & -C \\ 0 & \lambda I - D \end{bmatrix}\right) = det(\lambda I - B) det(\lambda I - D)$$

So the eigenvalues of A is equivalent to being an eigenvalue of B or D.

7.2.16 Characteristic polynomial is $\lambda^2 - (a+c)\lambda - ac + b^2$. So the discriminant is $(a+c)^2 - 4(ac-b^2) = (a-c)^2 + 4b^2 > 0$. So there are two real roots.

7.2.20

$$(\lambda - \lambda_1)(\lambda - \lambda_2) = \lambda^2 - (lambda_1 + \lambda_2)\lambda + \lambda_1\lambda_2$$

$$det\left(\lambda I - \begin{bmatrix} a & b \\ c & d \end{bmatrix}\right) = \lambda^2 - (a+d)\lambda + (ad - bc)$$

So $\lambda_1 + \lambda_2 = a + d = tr(A)$.

7.2.24 eigenvalues are $\frac{1}{4}$ and 1.

7.3.6 eigenvalues are $\frac{7}{2} \pm \frac{\sqrt{57}}{2}$. eigen basis is $\begin{bmatrix} 3 \\ \frac{3}{2} \pm \frac{\sqrt{57}}{2} \end{bmatrix}$.

7.3.20 The eigenspace for 1 is the the kernel of $\begin{bmatrix} 0 & a & b \\ 0 & 0 & c \\ 0 & 0 & 1 \end{bmatrix}$ which row reduces to $\begin{bmatrix} 0 & a & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}$. So the eigenspace has dimension 1 unless $a = 0$ and then it has dimension 2. The eigenspace for 2 is the kernel of $\begin{bmatrix} 1 & -a & -b \\ 0 & 1 & -c \\ 0 & 0 & 0 \end{bmatrix}$ which is always 1 dimensional.

7.3.24

$$\begin{bmatrix} 2 & 1 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 2 & 1 \\ 1 & 1 \end{bmatrix}^{-1} = \begin{bmatrix} -1 & 4 \\ -1 & 3 \end{bmatrix}$$

7.3.34
(a) From $SB = AS$ and $B\overline{x} = \overline{0}$, we get $A(S\overline{x}) = SB\overline{x} = \overline{0}$.
(b) Same as (a) with B and A switched and S and S^{-1} switched.
(c) Since $null(A) = null(B)$ and A and B are the same size, $rk(A) = rk(B)$.

7.4.4 The eigenvalues are 7 with eigenvector $\begin{bmatrix} 1 \\ 3 \end{bmatrix}$ and 0 with eigenvector $\begin{bmatrix} 2 \\ -1 \end{bmatrix}$. So $A = SDS^{-1}$ where $D = \begin{bmatrix} 7 & 0 \\ 0 & 0 \end{bmatrix}$ and $S = \begin{bmatrix} 1 & 2 \\ 3 & -1 \end{bmatrix}$.

7.4.16 The eigenvalues are 1 with eigenvector $\begin{bmatrix} 0 \\ -1 \\ 0 \end{bmatrix}$ The eigenvalues are 2 with eigenvector $\begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}$ and 3 with eigenvector $\begin{bmatrix} 2 \\ 0 \\ 1 \end{bmatrix}$. So $A = SDS^{-1}$ where $D = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 3 \end{bmatrix}$ and $S = \begin{bmatrix} 0 & 1 & 2 \\ -1 & 0 & 0 \\ 0 & 1 & 1 \end{bmatrix}$.

7.4.22 If $b = 1$ and $a \neq 0$, then there is only one eigenspace and it is of dimension 1, so it is not diagonalizable. In all other cases it is diagonalizable.

7.2.40

$$T(y) = 5y' - 3y = \lambda y$$

$$y' - \frac{3-\lambda}{5} y = 0$$

$$y = ce^{(3-\lambda)t/5}$$

So every real number is an eigenvalue with an eigenspace of dimension 1.

7.4.46

$$T(x_0, x_1, \ldots) = (x_0, x_2, \ldots) = \lambda (x_0, x_1, \ldots)$$

The odd terms can be anything. For $i > 0$, $x_{2i} = \lambda x_i$. If $\lambda = 1$ then x_0 can be anything. Otherwise $x_0 = 0$.

7.4.56 Since $M = SNS^{-1}$,

$$\begin{aligned} \lambda^n det(\lambda I - AB) &= det(\lambda I - M) = det\left(\lambda I - SNS^{-1}\right) = det\left(S(\lambda I - N)S^{-1}\right) \\ &= det(S)det(\lambda I - N)det(S)^{-1} = det(\lambda I - N) = \lambda^n det(\lambda I - BA) \end{aligned}$$

So

$$det(\lambda I - AB) = det(\lambda I - AB)$$

7.4.62

$$T(y) = y'' + ay' + by = \lambda y$$

$$y'' + ay' + (b - \lambda)y = 0$$

This differential equation always has a solution space of dimension exactly 2. So every number is an eigenvalue with a 2-dimensional eigenspace.