SOLUTIONS: ASSIGNMENT 9

6.2.6 *Use Gaussian elimination to find the determinant of the matrix.*

$$
\begin{aligned}
\det \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & 4 & 4 \\ 1 & -1 & 2 & -2 \\ 1 & -1 & 8 & -8 \end{bmatrix} &= \det \begin{bmatrix} 1 & 1 & 1 & 1 \\ 0 & 0 & 3 & 3 \\ 0 & -2 & 1 & -3 \\ 0 & -2 & 7 & -9 \end{bmatrix} = \det \begin{bmatrix} 1 & 1 & 1 & 1 \\ 0 & 0 & 3 & 3 \\ 0 & -2 & 1 & -3 \\ 0 & 0 & 6 & 12 \end{bmatrix} \\
&= \det \begin{bmatrix} 1 & 1 & 1 & 1 \\ 0 & 0 & 3 & 3 \\ 0 & -2 & 1 & -3 \\ 0 & 0 & 0 & 6 \end{bmatrix} = \det \left[- \begin{matrix} 1 & 1 & 1 & 1 \\ 0 & -2 & 1 & -3 \\ 0 & 0 & 3 & 3 \\ 0 & 0 & 0 & 6 \end{matrix} \right] \\
&= -1 \cdot (-2) \cdot 3 \cdot 6 = 36
\end{aligned}
$$

6.2.12 Consider a 4×4 *matrix* A *with rows* $\vec{v_1}, \vec{v_2}, \vec{v_3}, \vec{v_4}$. *If* $\det(A) = 8$ *, find*

$$\det \begin{bmatrix} \vec{v_4} \\ \vec{v_2} \\ \vec{v_3} \\ \vec{v_1} \end{bmatrix}.$$

This matrix is obtained from A by a row swap, so its determinant is $\det(A) = -8$.

6.2.16 *Using the same row vectors as in* 6.2.12*,*

$$\det \begin{bmatrix} 6\vec{v_1} + 2\vec{v_4} \\ \vec{v_2} \\ \vec{v_3} \\ 3\vec{v_1} + \vec{v_4} \end{bmatrix}.$$

The rows of this matrix are linearly dependent, because $2(3\vec{v_1} + \vec{v_4}) = 6\vec{v_1} + \vec{v_4}$. A matrix whose rows are linearly dependent must have determinant 0.

6.2.18 *Find the determinant of the linear transformation* $T(f(t)) = f(3t-2)$ *from* P_2 *to* P_2.

A basis for P_2 is $\{1, t, t^2\}$. Applying T to each basis element, we have $T(1) = 1$, $T(t) = 3t - 2$, and $T(t^2) = (3t-2)^2 = 9t^2 - 12t + 4$. The matrix of T with respect to this basis is

$$\begin{bmatrix} 1 & -2 & 4 \\ 0 & 3 & -12 \\ 0 & 0 & 9 \end{bmatrix},$$

which has a determinant of 27.

6.2.22 *Find the determinant of the linear transformation* $T(f(t)) = f(-t)$ *from* P_n *to* P_n.

A basis for P_n is $\{t^k : 0 \leq k \leq\}$. If k is odd, then $T(t^k) = -t^k$, and if k is even then $T(t^k) = t^k$. Therefore the determinant of T is $(-1)^m$ where m is the number of odd numbers between 0 and n, inclusive. Let N be the remainder when n is divided by 4. A quick check shows that m is even if N is 0 or 3, while m is odd if N is 1 or 2. The quantity $\dfrac{n(n+1)}{2}$ is also even if N is 0 or 3 (because then either n or $n+1$ is divisible by 4) but is even if N is 1 or 2. Thus the determinant of T is $(-1)^{\dfrac{n(n+1)}{2}}$.

6.2.30 *Consider two distinct numbers* a *and* b. *We define the function*

$$f(t) = \det \begin{bmatrix} 1 & 1 & 1 \\ a & b & t \\ a^2 & b^2 & t^2 \end{bmatrix}.$$

(a) *Show that* $f(t)$ *is a quadratic function. What is the coefficient of* t^2*?*

Expanding down the third column, we have $f(t) = D_1 - D_2 t + D_3 t^2$, where the D_i are determinants of 2×2 matrices that contain no factor of t. Since $D_3 = b - a$, $f(t)$ is a quadratic function with leading coefficient $b - a$.

(b) *Explain why* $f(a) = f(b) = 0$. *Conclude that* $f(t) = k(t-a)(t-b)$, *for some constant* k. *Find* k, *using your work in part (a).*

If $t = a$, the first and third columns of the matrix are the same, so it has determinant 0. Likewise, if $t = b$, the second and third columns of the matrix are the same. This shows that $f(a) = f(b) = 0$. It follows that $f(t)$ has the factors $t - a$ and $t - b$; since $f(t)$ is quadratic, it can have no other non-constant factors, so $f(t) = k(t-a)(t-b)$ for some constant k. This constant is equal to the leading coefficient of $f(t)$ which is $b-a$ by part (a).

(c) *For which values of* t *is the matrix invertible?*

A matrix is invertible if and only if its determinant is nonzero. In part (b) we found that $f(t) = (b-a)(t-a)(t-b)$, which is nonzero for all values of t except a and b.

6.2.31 Vandermonde determinants: Consider distinct scalars $a_0, a_1, \cdots, a_n$. We define the $(n+1) \times (n+1)$ matrix

$$A = \begin{bmatrix} 1 & 1 & \cdots & 1 \\ a_0 & a_1 & \cdots & a_n \\ a_0^2 & a_1^2 & \cdots & a_n^2 \\ \vdots & \vdots & & \vdots \\ a_0^n & a_1^n & \cdots & a_n^n \end{bmatrix}.$$

Vandermonde showed that

$$\det(A) = \prod_{i>j}(a_i - a_j),$$

the product of all differences $a_i - a_j$ where i exceeds j.

(a) Verify this formula in the case of $n = 1$.

If $n = 1$ then $A = \begin{bmatrix} 1 & 1 \\ a_0 & a_1 \end{bmatrix}$ and $\det(A) = a_1 - a_0$

(b) *Suppose the Vandermonde formula holds for $n-1$. You are asked to demonstrate it for n. Consider the function*

$$f(t) = \det \begin{bmatrix} 1 & 1 & \cdots & 1 & 1 \\ a_0 & a_1 & \cdots & a_{n-1} & t \\ a_0^2 & a_1^2 & \cdots & a_{n-1}^2 & t^2 \\ \vdots & \vdots & & \vdots & \vdots \\ a_0^n & a_1^n & \cdots & a_{n-1}^n & t^n \end{bmatrix}$$

Explain why $f(t)$ is a polynomial of nth degree. Find the coefficient k of t^n using Vandermonde's formula for $n-1$. Explain why

$$f(a_0) = f(a_1) = \cdots = f(a_{n-1}) = 0.$$

Conclude that

$$f(t) = k(t - a_0)(t - a_1)\cdots(t - a_{n-1}).$$

for the scalar k you found above. Substitute $t = a_n$ to demonstrate Vandermonde's formula.

Expanding down the rightmost column of the matrix, we have

$$f(t) = \pm D_0 \mp D_1 t \pm \cdots + D_n t^n$$

where the D_i are determinants of $n \times n$ matrices that contain no factor of t. Since D_n is the Vandermonde determinant of the $n \times n$ matrix with coefficients a_0 through a_{n-1}, we conclude that $f(t)$ is an nth degree polynomial with leading coefficient

$$k = \prod_{n>i>j} (a_i - a_j).$$

If $t = a_0$, then $f(t) = 0$ because the leftmost and rightmost columns of the matrix are equal. Applying the same reasoning to a_1 through a_{n-1}, we can see that

$$f(a_0) = f(a_1) = \cdots = f(a_{n-1}) = 0.$$

Since the n values a_i for $0 \leq i \leq n$ are all distinct, and $f(t)$ is an nth degree polynomial, it must be the case that

$$f(t) = k(t - a_0)(t - a_1) \cdots (t - a_{n-1})$$

for the value of k found above. As stated, substituting $t = a_n$ into this equation demonstrates Vandermonde's formula for n.

6.3.2 Find the area of the triangle defined by $\begin{bmatrix} 3 \\ 7 \end{bmatrix}$ *and* $\begin{bmatrix} 8 \\ 2 \end{bmatrix}$.

The area of the triangle formed by two vectors is half the area of the parallelogram formed by the same two vectors. Thus, the area is

$$\frac{1}{2} \det \begin{bmatrix} 8 & 3 \\ 2 & 7 \end{bmatrix} = \frac{1}{2}(56 - 6) = 25.$$

6.3.4 *Consider the area A of a triangle with vertices* $\begin{bmatrix} a_1 \\ a_2 \end{bmatrix}, \begin{bmatrix} b_1 \\ b_2 \end{bmatrix}, \begin{bmatrix} c_1 \\ c_2 \end{bmatrix}$. *Express A in terms of*

$$D = \det \begin{bmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ 1 & 1 & 1 \end{bmatrix}.$$

The answer is $A = \frac{|D|}{2}$. On one hand, note that A is the same as the area of the triangle with vertices $\begin{bmatrix} 0 \\ 0 \end{bmatrix}, \begin{bmatrix} b_1 - a_1 \\ b_2 - a_2 \end{bmatrix}, \begin{bmatrix} c_1 - a_1 \\ c_2 - a_2 \end{bmatrix}$. This area is half the absolute value of the quantity

$$\det \begin{bmatrix} b_1 - a_1 & c_1 - a_1 \\ b_2 - a_2 & c_2 - a_2 \end{bmatrix} = (b_1 - a_1)(c_2 - a_2) - (c_1 - a_1)(b_2 - a_2).$$

On the other hand,

$$\begin{aligned} D &= (a_1b_2 - b_1a_2) + (c_1a_2 - a_1c_2) + (b_1c_2 - c_1b_2) \\ &= (b_1 - a_1)c_2 - a_1c_2 + (a_1 - c_1)b_2 + a_1b_2 \\ &= (b_1 - a_1)(c_2 - a_2) - (c_1 - a_1)(b_2 - a_2) \end{aligned}$$

where $0 = a_1a_2 - a_1a_2$ was added to the equation in the last step.

6.3.10 *Consider an $n \times n$ matrix $A = [\ \vec{v_1} \quad \vec{v_2} \quad \cdots \quad \vec{v_n}\ ]$. What is the relationship between the product $\|\vec{v_1}\|\|\vec{v_2}\|\cdots\|\vec{v_n}\|$ and det(A)? When is*

$$\det(A) = \|\tilde{v_1}\|\|\tilde{v_2}\|\cdots\|\tilde{v_n}\|?$$

Using a fact proven in the textbook (Fact 6.3.4),

$$|\det(A)| = \|\tilde{v_1}\|\|\tilde{v_2}^{\perp}\|\cdots\|\tilde{v_n}^{\perp}\| =\leq \|\tilde{v_1}\|\|\tilde{v_2}\|\cdots\|\tilde{v_n}\|$$

with equality if and only if $\|\vec{v_i}^{\perp}\| = \|\vec{v_i}\|$ for all i; that is, if the v_i form an orthogonal basis for $\mathbb{R}^n$.

6.3.26 *Consider an $n \times n$ matrix A with integer entries such that det $(A) = 1$. Are the entries of A^{-1} necessarily integers? Explain.*

Yes, A^{-1} must have integer entries. Since $\det(A) = 1$, $A^{-1} = \text{adj}(A)$. But $\text{adj}(A)$ has integer entries because A does.

7.1.2 *Let A be an invertible $n \times n$ matrix and $\vec{v}$ an eigenvector of A with associated eigenvalue λ. Is $\vec{v}$ an eigenvector of A^{-1}? If so, what is the eigenvalue?*

We can see that $\vec{v}$ is an eigenvector of A^{-1} when we multiply the equation $A\vec{v} = \lambda v$ by A^{-1}, as follows:

$$A^{-1}\vec{v} = \frac{1}{\lambda}A^{-1}(\lambda\vec{v}) = \frac{1}{\lambda}A^{-1}(A\vec{v}) = \frac{1}{\lambda}\vec{v},$$

so the eigenvalue is $\dfrac{1}{\lambda}$.

7.1.3 *Let A be an invertible $n \times n$ matrix and $\vec{v}$ an eigenvector of A with associated eigenvalue λ. Is $\vec{v}$ an eigenvector of $A + 2I_n$? If so, what is the eigenvalue?*

Yes, because

$$(A + 2I_n)(\vec{v}) = A\vec{v} + 2I_n\vec{v} = \lambda\vec{v} + 2\vec{v} = (\lambda + 2)\vec{v},$$

so the eigenvalue is $\lambda + 2$.

7.1.5 *If a vector $\vec{v}$ is an eigenvector of both A and B, is $\vec{v}$ necessarily an eigenvector of $A+B$?*

Yes. If the eigenvalues for A and B are α and β, respectively, then

$$(A+B)(\vec{v}) = A\vec{v} + B\vec{v} = \alpha\vec{v} + \beta\vec{v} = (\alpha+\beta)\vec{v},$$

so the eigenvalue for $A+B$ is $\alpha+\beta$.

7.1.6 *If a vector $\vec{v}$ is an eigenvector of both A and B, is $\vec{v}$ necessarily an eigenvector of AB?*

Yes. If the eigenvalues for A and B are α and β, respectively, then

$$(AB)(\vec{v}) = A(B\vec{v}) = A(\beta\vec{v}) = \beta(A\vec{v}) = \beta\alpha\vec{v} = \alpha\beta\vec{v},$$

so the eigenvalue for AB is $\alpha\beta$.

7.1.16 *Arguing geometrically, find all eigenvectors and eigenvalues of the linear transformation that represents rotation by* 180 *degrees in* $\mathbb{R}^2$. *Find a basis of eigenvectors.*

If T is the name of this transformation, then $T\vec{v} = -\vec{v}$ for every $\vec{v}$ in $\mathbb{R}^2$. Every vector in $\mathbb{R}^2$ is an eigenvector of T with eigenvalue -1. The standard basis for $\mathbb{R}^2$ is a basis of eigenvectors, for example.

7.1.34 *Suppose $\vec{v}$ is an eigenvector of the $n \times n$ matrix A, with eigenvalue* 4. *Explain why $\vec{v}$ is an eigenvector of $A^2 + 2A + 3I_n$. What is the associated eigenvalue?*

By repeatedly using the equation $A\vec{v} = 4\vec{v}$, we find that

$$(A^2 + 2A + 3I_n)(\vec{v}) = A^2\vec{v} + 2A\vec{v} + 3I_n\vec{v} = 16\vec{v} + 2 \cdot 4\vec{v} + 3\vec{v} = 27\vec{v},$$

so the associated eigenvalue is 27.

7.1.36 *Find a 2×2 matrix A such that $\begin{bmatrix} 3 \\ 1 \end{bmatrix}$ and $\begin{bmatrix} 1 \\ 2 \end{bmatrix}$ are eigenvectors of A, with eigenvalues* 5 *and* 10, *respectively.*

Write $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$. We obtain the equations

$$\begin{aligned} 3a + b &= 15 \\ 3c + d &= 5 \\ a + 2b &= 10 \\ c + 2d &= 20 \end{aligned}$$

The solution to this system is $a = 4, b = 3, c = -2, d = 11$.

7.1.44 *For $m \leq n$, find the dimension of the space V of all $n \times n$ matrices A for which all the vectors $\vec{e_1}, \cdots, \vec{e_n}$ are eigenvectors.*

The dimension of V is $m + n(n - m)$. Suppose A is a matrix in V. If the corresponding eigenvalues are $\lambda_1, \cdots, \lambda_m$ then the first m columns of A are $\lambda_1 \vec{e_1}, \cdots, \lambda_m \vec{e_m}$. Each of these eigenvalues represents one free variable. The remaining $n - m$ columns can be arbitrary vectors in $\mathbb{R}^n$, for a total of $n(n-m)$ free variables. The number of free variables for the entire matrix is thus $m + n(n - m)$.